On the constants in a basic inequality for the Euler and Navier-Stokes equations

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Abstract

We consider the incompressible Euler or Navier-Stokes (NS) equations on a d-dimensional torus $\mathbf{T}^d$; the quadratic term in these equations arises from the bilinear map sending two velocity fields $v, w : \mathbf{T}^d \to \mathbf{R}^d$ into $v \bullet \partial w$, and also involves the Leray projection $\mathfrak{L}$ onto the space of divergence free vector fields. We derive upper and lower bounds for the constants in some inequalities related to the above quadratic term; these bounds hold, in particular, for the sharp constants $K_{nd} \equiv K_n$ in the basic inequality $\|\mathfrak{L}(v \bullet \partial w)\|_n \leqslant K_n \|v\|_n \|w\|_{n+1}$, where $n \in (d/2, +\infty)$ and v, w are in the Sobolev spaces $\mathbb{H}^n_{\Sigma 0}, \mathbb{H}^{n+1}_{\Sigma 0}$ of zero mean, divergence free vector fields of orders n and $n+1$, respectively. As examples, the numerical values of our upper and lower bounds are reported for $d = 3$ and some values of n. Some practical motivations are indicated for an accurate analysis of the constants K_n.

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1 Introduction

The incompressible Euler or Navier-Stokes (NS) equations in d space dimensions can be written as

$$\frac{\partial u}{\partial t} = -\mathfrak{L}(u \bullet \partial u) + \nu \Delta u + f \ , \tag{1.1}$$

where: $u = u(x, t)$ is the divergence free velocity field; $x = (x_s)_{s=1,...,d}$ are the space coordinates (yielding the derivatives $\partial_s := \partial/\partial x_s$); $\Delta := \sum_{s=1}^{d} \partial_{ss}$ is the Laplacian; $(u \bullet \partial u)_r := \sum_{s=1}^{d} u_s \partial_s u_r$ $(r = 1, ..., d)$; $\mathfrak{L}$ is the Leray projection onto the space of divergence free vector fields; $\nu = 0$ for the Euler equations; $\nu \in (0, +\infty)$ (in fact $\nu = 1$, after rescaling) for the NS equations; $f = f(x, t)$ is the Leray projected density of external forces.
In this paper we stick to the case of space periodic boundary conditions; so, x ranges in the d-dimensional torus $\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$. As well known, for any solution u of Eqs. (1.1), the (spatial) mean $\langle u \rangle := (2\pi)^{-d} \int_{\mathbf{T}^d} u \, dx$ evolves according to $d\langle u \rangle/dt = \langle f \rangle$, and the zero mean vector field $u - \langle u \rangle$ fulfills an equation like (1.1), with f replaced by a new, zero mean forcing term (see, e.g., [7]); due to these remarks, the analysis of Eq. (1.1) can be reduced to the case where $\langle u \rangle = 0$, $\langle f \rangle = 0$.
Our functional setting for the incompressible Euler/NS equations relies on H^n Sobolev spaces. More precisely we consider, for suitable (integer or noninteger) values of n, the spaces

$$\mathbb{H}^n_0(\mathbf{T}^d) \equiv \mathbb{H}^n_0 := \{ v : \mathbf{T}^d \to \mathbf{R}^d \mid \sqrt{-\Delta}^{\,n} v \in \mathbb{L}^2(\mathbf{T}^d), \ \langle v \rangle = 0 \} \ , \tag{1.2}$$

$$\mathbb{H}^n_{\Sigma 0}(\mathbf{T}^d) \equiv \mathbb{H}^n_{\Sigma 0} := \{ v \in \mathbb{H}^n_0 \mid \text{div } v = 0 \} \tag{1.3}$$

(the subscripts 0, Σ recall the vanishing of the mean and of the divergence, respectively). For each n, we equip $\mathbb{H}^n_0$ with the standard inner product and the norm

$$\langle v | w \rangle_n := \langle \sqrt{-\Delta}^{\,n} v | \sqrt{-\Delta}^{\,n} w \rangle_{L^2} \ , \qquad \| v \|_n := \sqrt{\langle v | v \rangle_n} \ , \tag{1.4}$$

which can be restricted to the (closed) subspace $\mathbb{H}^n_{\Sigma 0}$. We can now pass to discuss Eq. (1.1) with $u(\cdot, t) \in \mathbb{H}^n_{\Sigma 0}$ for each t.
A fully quantitative treatment of several problems related to the above functional setting (such as estimates on the time of existence of the solution of (1.1) for a given datum, estimates on its distance from any approximate solution, etc.) relies on the constants in some inequalities about the bilinear map sending two vector fields v, w on $\mathbf{T}^d$ into $v \bullet \partial w$, or about the composition of this map with $\mathfrak{L}$. Here, we wish to analyze the constants in some inequality of this kind.
To describe precisely the contents of this paper, let us recall that the assumptions $n > d/2$, $v \in \mathbb{H}^n_{\Sigma 0}$ and $w \in \mathbb{H}^{n+1}_0$ imply $v \bullet \partial w \in \mathbb{H}^n_0$, whence $\mathfrak{L}(v \bullet \partial w) \in \mathbb{H}^n_{\Sigma 0}$. In this paper we consider the basic inequality

$$\| \mathfrak{L}(v \bullet \partial w) \|_n \leqslant K_n \| v \|_n \| w \|_{n+1} \qquad \text{for } n \in (\frac{d}{2}, +\infty), \ v \in \mathbb{H}^n_{\Sigma 0}, \ w \in \mathbb{H}^{n+1}_{\Sigma 0} \ ; \tag{1.5}$$

our aim is to give quantitative upper and lower bounds on the sharp constant $K_n \equiv K_{nd}$ appearing therein. We use the fact that $K_n \leqslant K'_n$, where K'_n is the sharp constant in the (auxiliary) inequality

$$\|v{\scriptstyle\bullet}\partial w\|_n \leqslant K'_n \|v\|_n \|w\|_{n+1} \qquad \text{for } n \in (\frac{d}{2}, +\infty),\ v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_0 \ . \tag{1.6}$$

Even though Eqs. (1.5) (1.6) are well known, little information can be found in the literature about the numerical values of the constants therein. Our approach produces fully computable upper and lower bounds $K^{\pm}_n \equiv K^{\pm}_{nd}$ such that

$$K^-_n \leqslant K_n \leqslant K'_n \leqslant K^+_n \tag{1.7}$$

for all $n > d/2$. As examples, the bounds $K^{\pm}_n$ are computed in dimension $d = 3$, for some values of n. In these cases the upper and lower bounds are reasonably close, at least for the purpose to apply them to the Euler/NS equations.
In a companion paper [9], we will propose upper and lower bounds for the constants $G_{nd} \equiv G_n$ in the inequality

$$|\langle v{\scriptstyle\bullet}\partial w | w\rangle_n| \leqslant G_n \|v\|_n \|w\|^2_n \quad \text{for } n \in (\frac{d}{2} + 1, +\infty),\ v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_{\Sigma 0} \ , \tag{1.8}$$

dating back to a seminal paper by Kato [3]; even in this case, very little is known on the values of the G_n's.
Let us exemplify a framework in which one could use the inequalities (1.5) (1.8) and their constants K_n, G_n; to this purpose we mention a result of Chernyshenko, Constantin, Robinson and Titi [2], that can be stated as follows. Consider the Euler/NS equation (1.1) with a specified initial condition $u(x, 0) = u_0(x)$; let $u_{ap} : \mathbf{T}^d \times [0, T_{ap}] \to \mathbf{R}^d$ be an approximate solution of this Cauchy problem with errors $\epsilon : \mathbf{T}^d \times [0, T_{ap}] \to \mathbf{R}^d$ on the equation and $\epsilon_0 : \mathbf{T}^d \to \mathbf{R}$ on the initial condition, by which we mean that

$$\epsilon := \frac{\partial u_{ap}}{\partial t} + \mathfrak{L}(u_{ap}{\scriptstyle\bullet}\partial u_{ap}) - \nu \Delta u_{ap} - f \ , \qquad \epsilon_0 := u_{ap}(\cdot, 0) - u_0 \ . \tag{1.9}$$

Fix $n \in (d/2 + 1, +\infty)$; then, Eq. (1.1) with datum u_0 has a (strong) exact solution u in $\mathbb{H}^n_{\Sigma 0}$ on a time interval $[0, T] \subset [0, T_{ap}]$ if T and u_{ap} fulfill the inequality

$$\|\epsilon_0\|_n + \int_0^T \|\epsilon(t)\|_n dt < \frac{1}{G_n T}\, e^{-\displaystyle\int_0^T (G_n \|u_{ap}(t)\|_n + K_n \|u_{ap}(t)\|_{n+1}) dt} \tag{1.10}$$

($u_{ap}(t) := u_{ap}(\cdot, t)$, $\epsilon(t) := \epsilon(\cdot, t)$). For a given datum u_0, one can try a practical implementation of the above criterion after choosing a suitable u_{ap} (say, a Galerkin approximate solution). Of course, T can be evaluated via (1.10) only in the presence

of quantitative information on K_n and G_n; on the other hand, no information on these constants is provided in [2].
The bounds for K_n, G_n derived here and in [9] could be useful in the above framework. We plan to use these bounds in a forthcoming paper [10] devoted to the existence condition (1.10) and to some refinement of it, suited as well to get bounds on $\|u(t) - u_{ap}(t)\|_n$.

Relations to other works. In a previous paper of ours [7], we derived (fairly rough) upper bounds on the constants in a variant of the inequality (1.6), using an approach similar, but less refined than the one proposed here; in any case, this allowed to start a quantitative analysis of some approximation methods for the incompressible NS equations.
In the same spirit, in [8] we evaluated the constants $\tilde{K}_\omega$ in the inequality $\|v \bullet \partial w\|_{-\omega} \leqslant \tilde{K}_\omega \|v\|_1 \|w\|_1$ for $\omega \in (1/2, 1)$ and $v, w \in \mathbb{H}^1_{\Sigma 0}(\mathbf{T}^3)$. This allowed to prove that the NS equations with no external forcing (as in (1.1), with $\nu = 1$ and $f = 0$) have a global mild solution in $\mathbb{H}^1_{\Sigma 0}(\mathbf{T}^3)$ if the initial datum u_0 fulfills the bound $\|\mathrm{rot}\, u_0\|_{L^2} \leqslant 0.407$; in this way we improved the condition $\|\mathrm{rot}\, u_0\|_{L^2} \leqslant 0.00724$ derived in [12] by Robinson and Sadowski. (We are aware that the ultimate aim in investigating the three-dimensional NS equations would be to prove global existence for *all* sufficiently smooth initial data; however, due to the current status of such investigations one could be content with much more modest goals, such as the previously cited bounds.)
To conclude, and to place the paper within a wider framework, let us mention that a quantitative analysis of other bilinear maps in Sobolev spaces has been proposed in recent years and employed in a number of papers to obtain conditions of existence or error bounds on approximation methods for other evolutionary PDEs, such as the nonlinear heat or Schrödinger equations, and the short-pulse equation (see, e.g., [5] [6] [1] [11]). A fully quantitative viewpoint has been developed for other aspects of the same equations, or for other PDE's (see, e.g., [4] [13] [14]).

Organization of the paper. In Section 2 we fix our standards about Sobolev spaces on $\mathbf{T}^d$, and introduce in this framework the Leray projection $\mathfrak{L}$ and the bilinear map $v, w \mapsto v \bullet \partial w$.
Section 3 states the main results of the paper; here we present our upper and lower bounds $K_n^\pm$, fulfilling Eq.(1.7) (in any space dimension d); these are the subject of Propositions 3.7 and 3.8, respectively. A major character of this section is a positive function $\mathcal{K}_n$, defined on the space $\mathbf{Z}^d \setminus \{0\}$ of nonzero Fourier wave vectors, whose sup determines our upper bound K_n^+; at each point $k \in \mathbf{Z}^d \setminus \{0\}$, $\mathcal{K}_n(k)$ is a sum (of convolutional type) over $\mathbf{Z}^d \setminus \{0, k\}$. The lower bound K_n^- given in the same section is an elementary function of n (and d). As examples, the numerical values of the bounds $K_n^\pm$ are reported for $d = 3$ and $n = 2, 3, 4, 5, 10$ (see Eq. (3.23)).
Section 4 contains the proofs of the previously mentioned Propositions 3.7, 3.8.
Three appendices are devoted to the practical evaluation of the function $\mathcal{K}_n$ and

of the bounds K_n^+. Appendix A presents some preliminary notations and results. Appendix B contains the main theorem (Proposition B.1) about the evaluation of $\mathcal{K}_n$ and of its sup. Finally, in Appendix C we give details on the computation of $\mathcal{K}_n$ and K_n^+ for the previously mentioned cases $d = 3$, $n = 2, 3, 4, 5, 10$.
For all the numerical computations required in this paper, as well as for some lengthy symbolic manipulations, we have used systematically the software MATHEMATICA. Throughout the paper, an expression like $r = a.bcde...$ means the following: computation of the real number r via MATHEMATICA produces as an output $a.bcde$, followed by other digits not reported for brevity.

2 Sobolev spaces on $\mathbf{T}^d$, and the Euler/NS quadratic nonlinearity

In this section we summarize our standard definitions and notations about spaces of periodic functions and distributions, and their applications to the incompressible Euler or NS equations; we consider, especially, Sobolev spaces on the torus. These standards were already described in [8], with some more details.
Throughout the paper, we consider any space dimension

$$d \geqslant 2 \ ; \tag{2.1}$$

r, s are indices running from 1 to d. For $a = (a_r)$, $b = (b_r) \in \mathbf{C}^d$ we put

$$a \bullet b := \sum_{r=1}^{d} a_r \, b_r \ , \qquad |a| := \sqrt{\overline{a} \bullet a} \ , \tag{2.2}$$

where $\overline{a} := (\overline{a_r})$ is the complex conjugate of a. Hereafter we refer to the d-dimensional torus

$$\mathbf{T}^d := \underbrace{\mathbf{T} \times ... \times \mathbf{T}}_{d \text{ times}} \ , \qquad \mathbf{T} := \mathbf{R}/(2\pi\mathbf{Z}) \ , \tag{2.3}$$

whose elements are typically written $x = (x_r)_{r=1,...d}$.

Distributions on $\mathbf{T}^d$, Fourier series and Sobolev spaces. We introduce the space of periodic distributions $D'(\mathbf{T}^d, \mathbf{C}) \equiv D'_{\mathbf{C}}$, which is the (topological) dual of $C^\infty(\mathbf{T}^d, \mathbf{C}) \equiv C^\infty_{\mathbf{C}}$; $\langle v, f \rangle \in \mathbf{C}$ denotes the action of a distribution $v \in D'_{\mathbf{C}}$ on a test function $f \in C^\infty_{\mathbf{C}}$.
We also consider the lattice $\mathbf{Z}^d$ of elements $k = (k_r)_{r=1,...,d}$. Each $v \in D'_{\mathbf{C}}$ has a unique (weakly convergent) Fourier series expansion

$$v = \sum_{k \in \mathbf{Z}^d} v_k e_k \ , \quad e_k(x) := \frac{1}{(2\pi)^{d/2}} \, e^{ik \bullet x} \text{ for } x \in \mathbf{T}^d \ , \quad v_k := \langle v, e_{-k} \rangle \ \in \mathbf{C} \ . \tag{2.4}$$

The complex conjugate of a distribution $v \in D'_{\mathbf{C}}$ is the unique distribution $\overline{v}$ such that $\overline{\langle v, f\rangle} = \langle \overline{v}, \overline{f}\rangle$ for each $f \in C^{\infty}_{\mathbf{C}}$; one has $\overline{v} = \sum_{k \in \mathbf{Z}^d} \overline{v_k}\, e_{-k}$.
The *mean* of $v \in D'_{\mathbf{C}}$ and the space of *zero mean* distributions are

$$\langle v \rangle := \frac{1}{(2\pi)^d}\langle v, 1\rangle = \frac{1}{(2\pi)^{d/2}} v_0 \ , \qquad D'_{\mathbf{C}0} := \{ v \in D'_{\mathbf{C}} \mid \langle v \rangle = 0 \} \tag{2.5}$$

(of course, $\langle v, 1\rangle = \int_{\mathbf{T}^d} v\, dx$ if $v \in L^1(\mathbf{T}^d, \mathbf{C}, dx)$). The relevant Fourier coefficients of zero mean distributions are labeled by the set

$$\mathbf{Z}^d_0 := \mathbf{Z}^d \setminus \{0\} \ . \tag{2.6}$$

The distributional derivatives $\partial/\partial x_s \equiv \partial_s$ and the Laplacian $\Delta := \sum_{s=1}^d \partial_{ss}$ send $D'_{\mathbf{C}}$ into $D'_{\mathbf{C}0}$ and, for each v, $\partial_s v = i \sum_{k \in \mathbf{Z}^d_0} k_s v_k e_k$, $\Delta v = -\sum_{k \in \mathbf{Z}^d_0} |k|^2 v_k e_k$. For any $n \in \mathbf{R}$, we further define

$$\sqrt{-\Delta}^{\,n} : D'_{\mathbf{C}} \to D'_{\mathbf{C}0} \ , \qquad v \mapsto \sqrt{-\Delta}^{\,n} v := \sum_{k \in \mathbf{Z}^d_0} |k|^n v_k e_k \ . \tag{2.7}$$

The space of *real* distributions is

$$D'(\mathbf{T}^d, \mathbf{R}) \equiv D' := \{ v \in D'_{\mathbf{C}} \mid \overline{v} = v \} = \{ v \in D'_{\mathbf{C}} \mid \overline{v_k} = v_{-k} \text{ for all } k \in \mathbf{Z}^d \} \ . \tag{2.8}$$

For $p \in [1, +\infty]$ we often consider the real space

$$L^p(\mathbf{T}^d, \mathbf{R}, dx) \equiv L^p \ , \tag{2.9}$$

mainly for $p = 2$. L^2 is a Hilbert space with the inner product $\langle v | w \rangle_{L^2} := \int_{\mathbf{T}^d} v(x) w(x) dx = \sum_{k \in \mathbf{Z}^d} \overline{v_k} w_k$ and the induced norm $\| v \|_{L^2} = \sqrt{\int_{\mathbf{T}^d} v^2(x) dx} = \sqrt{\sum_{k \in \mathbf{Z}^d} |v_k|^2}$.
The zero mean parts of D' and L^p are

$$D'_0 := \{ v \in D' \mid \ \langle v \rangle = 0 \} \ , \qquad L^p_0 := L^p \cap \mathbb{D}'_0 \ ; \tag{2.10}$$

all the above mentioned differential operators send D' into D'_0.
For each $n \in \mathbf{R}$, the *zero mean Sobolev space* $H^n_0(\mathbf{T}^d, \mathbf{R}) \equiv H^n_0$ is defined by

$$H^n_0 := \{ v \in D'_0 \mid \sqrt{-\Delta}^{\,n} v \in L^2 \} = \{ v \in D'_0 \mid \sum_{k \in \mathbf{Z}^d_0} |k|^{2n} |v_k|^2 < +\infty \ \} \ ; \tag{2.11}$$

this is a real Hilbert space with the inner product $\langle v | w \rangle_n := \langle \sqrt{-\Delta}^{\,n} v \mid \sqrt{-\Delta}^{\,n} w \rangle_{L^2} = \sum_{k \in \mathbf{Z}^d_0} |k|^{2n}\, \overline{v_k} w_k$ and the induced norm $\| v \|_n = \| \sqrt{-\Delta}^{\,n}\ v \|_{L^2} = \sqrt{\sum_{k \in \mathbf{Z}^d_0} |k|^{2n} |v_k|^2}$.
As a special case, if n is a *nonnegative integer* one proves that

$$H^n_0 = \{ v \in D'_0 \mid \partial_{s_1 \ldots s_n} v \in L^2 \ \ \forall s_1, \ldots, s_n \in \{1, \ldots., d\} \} \tag{2.12}$$

and that, for each v in the above space, $\|v\|_n = \sqrt{\sum_{s_1,\dots,s_n=1}^{d} \|\partial_{s_1\dots s_n} v\|_{L^2}^2}$.

Spaces of vector valued functions on $\mathbf{T}^d$. If $V(\mathbf{T}^d, \mathbf{R}) \equiv V$ is any vector space of *real* functions or distributions on $\mathbf{T}^d$, we write

$$\mathbb{V}(\mathbf{T}^d) \equiv \mathbb{V} := \{v = (v^1, ..., v^d) \mid v_r \in V \quad \text{for all } r\} . \tag{2.13}$$

In this way we can define, e.g., the spaces $\mathbb{D}'(\mathbf{T}^d) \equiv \mathbb{D}'$, $\mathbb{L}^p(\mathbf{T}^d) \equiv \mathbb{L}^p$ $(p \in [1, +\infty])$, $\mathbb{H}^n_0(\mathbf{T}^d) \equiv \mathbb{H}^n_0$. Any $v = (v_r) \in \mathbb{D}'$ is referred to as a (distributional) *vector field* on $\mathbf{T}^d$. We note that v has a unique Fourier series expansion (2.4) with coefficients

$$v_k := (v_{rk})_{r=1,\dots,d} \in \mathbf{C}^d , \qquad v_{rk} := \langle v_r, e_{-k}\rangle ; \tag{2.14}$$

as in the scalar case, the reality of v ensures $\overline{v_k} = v_{-k}$.
$\mathbb{L}^2$ is a real Hilbert space, with the inner product $\langle v|w\rangle_{L^2} := \int_{\mathbf{T}^d} v(x){\scriptstyle\bullet} w(x)dx = \sum_{k\in\mathbf{Z}^d} \overline{v_k}{\scriptstyle\bullet} w_k$ and the induced norm

$$\|v\|_{L^2} = \sqrt{\int_{\mathbf{T}^d} |v(x)|^2 dx} = \sqrt{\sum_{k\in\mathbf{Z}^d} |v_k|^2} . \tag{2.15}$$

We define componentwise the mean $\langle v\rangle \in \mathbf{R}^d$ of any $v \in \mathbb{D}'$ (see Eq. (2.5)); $\mathbb{D}'_0$ is the space of zero mean vector fields, and $\mathbb{L}^p_0 = \mathbb{L}^p \cap \mathbb{D}'_0$. We similarly define componentwise the operators $\partial_s, \Delta, \sqrt{-\Delta}^{\,n} : \mathbb{D}' \to \mathbb{D}'_0$.
For any real n, the n-th Sobolev space of zero mean vector fields $\mathbb{H}^n_0(\mathbf{T}^d) \equiv \mathbb{H}^n_0$ is made of all d-uples v with components $v_r \in H^n_0$; an equivalent definition can be given via Eq.(2.11), replacing therein L^2 with $\mathbb{L}^2$. $\mathbb{H}^n_0$ is a real Hilbert space with the inner product $\langle v|w\rangle_n := \langle \sqrt{-\Delta}^{\,n} v \,|\sqrt{-\Delta}^{\,n} w\rangle_{L^2} = \sum_{k\in\mathbf{Z}^d_0} |k|^{2n}\, \overline{v_k}{\scriptstyle\bullet}\, w_k$; the induced norm $\|\ \|_n$ is given by

$$\|v\|_n = \|\sqrt{-\Delta}^{\,n} v\|_{L^2} = \sqrt{\sum_{k\in\mathbf{Z}^d_0} |k|^{2n}\, |v_k|^2} . \tag{2.16}$$

Divergence free vector fields. Let div $: \mathbb{D}' \to D'_0$, $v \mapsto \operatorname{div} v := \sum_{r=1}^d \partial_r v_r = i \sum_{k\in\mathbf{Z}^d_0} (k{\scriptstyle\bullet}\, v_k) e_k$. Hereafter we introduce the space $\mathbb{D}'_\Sigma$ of *divergence free (or solenoidal) vector fields* and some subspaces of it, putting

$$\mathbb{D}'_\Sigma := \{v \in \mathbb{D}' \mid \operatorname{div} v = 0\} = \{v \in \mathbb{D}' \mid k{\scriptstyle\bullet}\, v_k = 0 \ \forall k \in \mathbf{Z}^d\ \} ; \tag{2.17}$$

$$\mathbb{D}'_{\Sigma 0} := \mathbb{D}'_\Sigma \cap \mathbb{D}'_0 , \quad \mathbb{L}^p_\Sigma := \mathbb{L}^p \cap \mathbb{D}'_\Sigma , \ \mathbb{L}^p_{\Sigma 0} := \mathbb{L}^p \cap \mathbb{D}'_{\Sigma 0} \quad (p \in [1, +\infty]) , \tag{2.18}$$

$$\mathbb{H}^n_{\Sigma 0} := \mathbb{D}'_\Sigma \cap \mathbb{H}^n_0 \quad (n \in \mathbf{R}). \tag{2.19}$$

$\mathbb{H}^n_{\Sigma 0}$ is a closed subspace of the Hilbert space $\mathbb{H}^n_{0}$, that we equip with the restrictions of $\langle \mid \rangle_n$, $\| \, \|_n$. The *Leray projection* is the (surjective) map

$$\mathfrak{L} : \mathbb{D}' \to \mathbb{D}'_{\Sigma} \ , \qquad v \mapsto \mathfrak{L} v := \sum_{k \in \mathbf{Z}^d} (\mathfrak{L}_k v_k) e_k \ , \tag{2.20}$$

where, for each k, $\mathfrak{L}_k$ is the orthogonal projection of $\mathbf{C}^d$ onto the orthogonal complement of k; more explicitly, if $c \in \mathbf{C}^d$,

$$\mathfrak{L}_0 c = c \ , \qquad \mathfrak{L}_k c = c - \frac{k \bullet c}{|k|^2} \, k \quad \text{for } k \in \mathbf{Z}^d_0 \ . \tag{2.21}$$

From the Fourier representations of $\mathfrak{L}$, $\langle \ \rangle$, etc., one easily infers that

$$\langle \mathfrak{L} v \rangle = \langle v \rangle \text{ for } v \in \mathbb{D}', \quad \mathfrak{L}\mathbb{D}'_0 = \mathbb{D}'_{\Sigma 0}, \quad \mathfrak{L}\mathbb{L}^2 = \mathbb{L}^2_{\Sigma}, \quad \mathfrak{L}\mathbb{H}^n_0 = \mathbb{H}^n_{\Sigma 0} \text{ for } n \in \mathbf{R} \ . \tag{2.22}$$

Furthermore, $\mathfrak{L}$ is an orthogonal projection in each one of the Hilbert spaces $\mathbb{L}^2$, $\mathbb{H}^n_0$; in particular,

$$\| \mathfrak{L} v \|_n \leqslant \| v \|_n \qquad \text{for } v \in \mathbb{H}^n_0 \ . \tag{2.23}$$

The quadratic Euler/NS nonlinearity. We are now ready to define precisely and to analyze the bilinear map sending two (sufficiently regular) vector fields v, w on $\mathbf{T}^d$ into $v \bullet \partial w$, and the composition of this map with $\mathfrak{L}$. Throughout this paragraph we assume

$$v \in \mathbb{L}^2 \ , \quad \partial_s w \in \mathbb{L}^2 \ \ (s = 1, ..., d) \ ; \tag{2.24}$$

the above condition on the derivatives of w implies $w \in \mathbb{L}^2$. The statements in the forthcoming two Lemmas are known, and proved only for completeness.

2.1 ***Lemma.*** *Consider the vector field $v \bullet \partial w$ on $\mathbf{T}^d$, of components*

$$(v \bullet \partial w)_r := \sum_{s=1}^{d} v_s \partial_s w_r \ ; \tag{2.25}$$

this is well defined and belongs to $\mathbb{L}^1$. With the additional assumption $\operatorname{div} v = 0$, one has $\langle v \bullet \partial w \rangle = 0$ (which also implies $\langle \mathfrak{L}(v \bullet \partial w) \rangle = 0$, see (2.22)).

Proof. Each component $(v \bullet \partial w)_r$, being a sum of products of L^2 functions, is evidently in L^1.

Now, assume provisionally that v, w are C^1; then $\int_{\mathbf{T}^d} (v \bullet \partial w)_r \, dx = \sum_{s=1}^d \int_{\mathbf{T}^d} v_s \partial_s w_r \, dx$ $= - \sum_{s=1}^d \int_{\mathbf{T}^d} (\partial_s v_s) w_r \, dx$ (integrating by parts), i.e.,

$$\int_{\mathbf{T}^d} (v \bullet \partial w)_r \, dx = - \int_{\mathbf{T}^d} (\operatorname{div} v) \, w_r \, dx \ . \tag{2.26}$$

By simple density arguments, (2.26) holds whenever $v, \partial_s w \in \mathbb{L}^2$ and $\operatorname{div} v \in L^2$. In particular, $\int_{\mathbf{T}^d} v \bullet \partial w \, dx = 0$ if $v, \partial_s w \in \mathbb{L}^2$ and $\operatorname{div} v = 0$. $\square$

2.2 ***Lemma.*** *$v{\bullet}\partial w$ has Fourier coefficients*

$$(v{\bullet}\,\partial w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d} [v_h{\bullet}\,(k-h)] w_{k-h} \qquad \textit{for all } k \in \mathbf{Z}^d \ . \tag{2.27}$$

Proof (Sketch). Consider the Fourier coefficients v_{rk}, w_{sk}; then $(\partial_r w_s)_k = ik_r\, w_{sk}$. The pointwise product corresponds to $(2\pi)^{-d/2}$ times the convolution of the Fourier coefficients; thus $(v{\bullet}\,\partial w)_{sk} = i(2\pi)^{-d/2} \sum_{r=1}^d \sum_{h\in\mathbf{Z}^d} v_{rh}(k-h)_r w_{s,k-h}$; the vector form of this statement is Eq. (2.27). □

To conclude, we note that Eqs. (2.20) (2.27) imply

$$[\mathfrak{L}(v{\bullet}\partial w)]_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbf{Z}^d} [v_h{\bullet}\,(k-h)] \mathfrak{L}_k w_{k-h} \qquad \text{for all } k \in \mathbf{Z}^d \ , \tag{2.28}$$

with $\mathfrak{L}_k$ as in (2.21).

3 The basic inequality for the Euler/NS quadratic nonlinearity

Throughout this section we assume ($d \in \{2, 3, ...\}$ and)

$$n \in (\frac{d}{2}, +\infty) \ . \tag{3.1}$$

Given two vector fields v, w on $\mathbf{T}^d$, we have already discussed $v{\bullet}\partial w$ under the conditions $v, \partial_s w \in \mathbb{L}^2$; here we consider the much stronger assumptions v in $\mathbb{H}^n_{\Sigma 0}$, w in $\mathbb{H}^{n+1}_0$ or $\mathbb{H}^{n+1}_{\Sigma 0}$.
The forthcoming Proposition 3.1 is well known, and presented here only for completeness; as a matter of fact, the quantitative analysis performed later will also give, as a byproduct, an alternative proof of this Proposition.

3.1 ***Proposition.*** *Let $v \in \mathbb{H}^n_{\Sigma 0}$, $w \in \mathbb{H}^{n+1}_0$; then,*

$$v{\bullet}\partial w \in \mathbb{H}^n_0 \ . \tag{3.2}$$

The map

$$\mathbb{H}^n_{\Sigma 0} \times \mathbb{H}^{n+1}_0 \to \mathbb{H}^n_0, \qquad (v, w) \mapsto v{\bullet}\partial w \tag{3.3}$$

is bilinear and continuous.

Of course, continuity of the above map is equivalent to the existence of a nonnegative constant K', such that $\|v{\bullet}\partial w\|_n \leqslant K' \|v\|_n \|w\|_{n+1}$ for v, w as in the previous Proposition. A similar inequality holds as well for $\mathfrak{L}(v{\bullet}\partial w) \in \mathbb{H}^n_{\Sigma 0}$, since $\|\mathfrak{L}(v{\bullet}\partial w)\|_n \leqslant \|v{\bullet}\partial w\|_n$.

So, we have the "auxiliary inequality" (1.6) and the "basic inequality" (1.5) of the Introduction; of course, the sharp constants appearing therein can be defined as follows.

3.2 ***Definition.*** *We put*

$$K'_{nd} \equiv K'_n \tag{3.4}$$

$$:= \min\{K' \in [0, +\infty) \mid \|v{\bullet}\partial w\|_n \leqslant K' \|v\|_n \|w\|_{n+1} \textit{ for all } v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_0\}\ ;$$

$$K_{nd} \equiv K_n \tag{3.5}$$

$$:= \min\{K \in [0, +\infty) \mid \|\mathfrak{L}(v{\bullet}\partial w)\|_n \leqslant K \|v\|_n \|w\|_{n+1} \textit{ for all } v \in \mathbb{H}^n_{\Sigma 0},\ w \in \mathbb{H}^{n+1}_{\Sigma 0}\}\ .$$

(Note that all w's in (3.5) are divergence free, a property not required in (3.4).)

The considerations after Proposition 3.1 ensure that

$$K_n \leqslant K'_n\ ; \tag{3.6}$$

in the rest of the section (which is its original part) we present computable upper and lower bounds on K'_n and K_n, respectively.
The upper bound requires a more lengthy analysis; the final result relies on a function $\mathcal{K}_{nd} \equiv \mathcal{K}_n$, appearing in the forthcoming Definition 3.5. Hereafter we introduce some auxiliary notations, required to build $\mathcal{K}_n$.

3.3 ***Definition.*** *(i) Here and in the sequel, the exterior power $\bigwedge^2 \mathbf{R}^d$ is identified with the space of real, skew-symmetric $d \times d$ matrices $A = (A_{rs})_{r,s,=1,\ldots,d}$; this is equipped with the operation of exterior product*

$$\wedge : \mathbf{R}^d \times \mathbf{R}^d \to \bigwedge^2 \mathbf{R}^d, \quad (p, q) \mapsto p \wedge q \ \ \textit{s.t.} \ \ (p \wedge q)_{rs} := p_r q_s - q_r p_s\ . \tag{3.7}$$

(ii) We equip the above space with the norm

$$|\ | : \bigwedge^2 \mathbf{R}^d \to [0, +\infty), \qquad A = (A_{rs}) \mapsto |A| := \sqrt{\frac{1}{2} \sum_{r,s=1}^{d} |A_{rs}|^2}\ . \tag{3.8}$$

The operation (3.7) is bilinear and skew-symmetric; when composed with the norm (3.8), it gives a mapping

$$\mathbf{R}^d \times \mathbf{R}^d \to [0, +\infty)\ , \qquad (p, q) \mapsto |p \wedge q|\ , \tag{3.9}$$

which has the following, well known properties.

3.4 ***Proposition.*** *Let p, q in $\mathbf{R}^d$. Then*

$$|p \wedge q| = \sqrt{|p|^2|q|^2 - (p\bullet q)^2} \; = |p||q| \sin\vartheta \; , \tag{3.10}$$

where $\vartheta \equiv \vartheta(p, q) \in [0; \pi]$ is the convex angle between p and q (defined arbitrarily, if $p = 0$ or $q = 0$). So, $|p \wedge q|$ is the area of the parallelogram of sides p,q and

$$|p \wedge q| \leqslant |p||q| \; . \tag{3.11}$$

Now we are ready to construct the function $\mathcal{K}_n$, a major character of the section.

3.5 ***Definition.*** *We put*

$$\mathbf{Z}^d_{0k} := \mathbf{Z}^d \setminus \{0, k\} \qquad \textit{for each } k \in \mathbf{Z}^d_0 \; ; \tag{3.12}$$

$$\mathcal{K}_{nd} \equiv \mathcal{K}_n : \mathbf{Z}^d_0 \to (0, +\infty), \quad k \mapsto \mathcal{K}_n(k) := |k|^{2n} \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}}. \tag{3.13}$$

3.6 Remarks. (i) The sum in (3.13) has nonnegative terms, so it exists in principle as an element of $[0, +\infty]$. However,

$$\sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} < +\infty \text{ for all } k \in \mathbf{Z}^d_0 \; ; \tag{3.14}$$

in fact, for fixed k and $h \to \infty$,

$$\frac{|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} = O\Big(\frac{1}{|h|^{4n+2}}\Big) \tag{3.15}$$

and, for each family $(s_h)_{h \in \mathbf{Z}^d}$ with elements in $[0, +\infty)$, the relation $s_h = O(1/|h|^\nu)$ with $\nu > d$ implies, as well known, $\sum_{h \in \mathbf{Z}^d} s_h < +\infty$. These considerations justify the claim $\mathcal{K}_n(k) \in (0, +\infty)$ in (3.13).
(ii) In Eq. (3.13), one can insert at will the identity $h \wedge k = h \wedge (k-h)$ (following from the bilinearity and skew-symmetry of $\wedge$), and the inequality $|h \wedge (k-h)| \leqslant |h||k-h|$; this will be occasionally done in the sequel.
(iii) Let $r \in \{1, ..., d\}$, and let σ be any permutation of $\{1, ..., d\}$; introduce the reflection operator R_r and the permutation operator P_σ defined by

$$R_r, P_\sigma : \mathbf{R}^d \to \mathbf{R}^d \; , \tag{3.16}$$

$$R_r(k_1, .., k_r, ..., k_d) := (k_1, ..., -k_r, ..., k_d) \; , \qquad P_\sigma(k_1, ..., k_d) := (k_{\sigma(1)}, ..., k_{\sigma(d)}) \; ;$$

these are orthogonal operators (with respect to the inner product $\bullet$ of $\mathbf{R}^d$), and send $\mathbf{Z}^d_0$ into itself. We note that

$$\mathcal{K}_n(R_r k) = \mathcal{K}_n(k) \; , \qquad \mathcal{K}_n(P_\sigma k) = \mathcal{K}_n(k) \qquad \text{for each } k \in \mathbf{Z}^d_0 \; ; \tag{3.17}$$

for example, the first equality is checked expressing $\mathcal{K}_n(R_r k)$ via the definition (3.13), making a change of variable $h \mapsto R_r h$ in the sum therein and noting that $|(R_r h) \wedge (R_r k)| = |h \wedge k|$, $|R_r h| = |h|$, $|R_r k| = |k|$, $|R_r k - R_r h| = |k - h|$. The verification of the second inequality (3.17) proceeds similarly.
Due to the symmetry properties (3.17), the computation of $\mathcal{K}_n(k)$ can always be reduced to the case $k_1 \geqslant k_2 \geqslant ... \geqslant k_d \geqslant 0$.
(iv) In Appendix B we prove that

$$\sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_n(k) < +\infty \ . \tag{3.18}$$

This appendix also gives tools for the practical evaluation of $\mathcal{K}_n$ and of its sup. □

Let us pass to the desired upper bound, which is the following.

3.7 ***Proposition.*** *The constant K'_n defined by (3.4) has the upper bound*

$$K'_n \leqslant K^+_n \ , \tag{3.19}$$

$$K^+_n := \frac{1}{(2\pi)^{d/2}} \sqrt{\sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_n(k)} \quad (\textit{or any upper approximant for this}). \tag{3.20}$$

Proof. See Section 4. □

Let us pass to the problem of finding a lower bound for K_n; this can be obtained directly from the tautological inequality $K_n \geqslant \|v{\bullet}\partial w\|_n / \|v\|_n \|w\|_{n+1}$, choosing for v, w some suitable trial functions. A very simple choice of v, w yields the following.

3.8 ***Proposition.*** *The constant K_n defined by (3.5) has the lower bound*

$$K_n \geqslant K^-_n \ , \tag{3.21}$$

$$K^-_n := \frac{2^{n/2}}{(2\pi)^{d/2}} U_d \quad (\textit{or any round down for this number}) \ , \tag{3.22}$$

$$U_d := \begin{cases} (2 - \sqrt{2})^{1/2} = 0.76536... & \textit{if } d = 2, \\ 1 & \textit{if } d \geqslant 3. \end{cases}$$

Proof. See Section 4. □

Putting together Eqs. (3.6) (3.19) (3.21) we obtain a chain of inequalities, anticipated in the Introduction,

$$K^-_n \leqslant K_n \leqslant K'_n \leqslant K^+_n \ ;$$

here, the bounds $K^{\pm}_n$ can be computed explicitly from their definitions (3.20) (3.22).

3.9 Examples. For $d = 3$ and $n = 2, 3, 4, 5, 10$, we can take

$$K_2^- = 0.126 \ , \quad K_2^+ = 0.335 \ ; \qquad K_3^- = 0.179 \ , \quad K_3^+ = 0.323 \ , \tag{3.23}$$

$$K_4^- = 0.253 \ , \ K_4^+ = 0.441 \ ; \quad K_5^- = 0.359 \ , \ K_5^+ = 0.510 \ ; \quad K_{10}^- = 2.03 \ , \ K_{10}^+ = 2.88 \ .$$

In the above, the K_n^- are obtained rounding down to three digits the number $2^{n/2}(2\pi)^{-3/2}$; the K_n^+ are obtained from upper approximation of the sup in (3.20), as illustrated in Appendix C. The ratios K_n^-/K_n^+ are 0.376..., 0.554..., 0.573..., 0.703..., 0.704... for $n = 2, 3, 4, 5, 10$, respectively.
To avoid misunderstandings related to these examples, we repeat that the approach of this paper applies as well to noninteger values of n.

4 Proof of Propositions (3.1 and) 3.7, 3.8

Throughout the section $n \in (d/2, +\infty)$.

4.1 ***Lemma.*** *Let*

$$p, q \in \mathbf{R}^d \setminus \{0\} \ , \quad z \in \mathbf{C}^d, \ p{\bullet}z = 0 \ , \tag{4.1}$$

and $\vartheta(p,q) \equiv \vartheta \in [0, \pi]$ *be the convex angle between* q *and* p*. Then*

$$|q{\bullet}z| \leqslant \sin\vartheta \, |q||z| = \frac{|p \wedge q|}{|p|} \, |z| \ . \tag{4.2}$$

Proof. We choose an orthonormal basis $(\eta_r)_{r=1,...,d}$ of $\mathbf{R}^d$ so that q be a positive multiple of η_1, p be in the span of η_1, η_2 and $p{\bullet}\eta_2 \geqslant 0$; then

$$q = |q|\eta_1 \ , \qquad p = |p|(\cos\vartheta \, \eta_1 + \sin\vartheta \, \eta_2) \ . \tag{4.3}$$

The $(d-1)$ vectors

$$-\sin\vartheta \, \eta_1 + \cos\vartheta \, \eta_2, \eta_3, ..., \eta_d \tag{4.4}$$

clearly form an orthonormal basis for

$$\{p\}^\perp := \{z \in \mathbf{C}^d \mid p{\bullet}z = 0\} \ ; \tag{4.5}$$

so, any $z \in \{p\}^\perp$ has a unique expansion

$$z = z^{(2)}(-\sin\vartheta \, \eta_1 + \cos\vartheta \, \eta_2) + z^{(3)}\eta_3 + ... + z^{(d)}\eta_d, \ z^{(t)} \in \mathbf{C} \text{ for } t = 2, ..., d \ . \tag{4.6}$$

From Eqs. (4.3) (4.6) we get

$$q{\bullet}z = -\sin\vartheta \, |q| z^{(2)} \ , \tag{4.7}$$

which implies

$$|q{\bullet}z| = \sin\vartheta\,|q||z^{(2)}| \leqslant \sin\vartheta|q||z| \ . \tag{4.8}$$

So, the inequality in (4.2) is proved; the subsequent equality in (4.2) follows from (3.10). □

Proofs of Propositions 3.1 and 3.7. We choose $v \in \mathbb{H}^n_{\Sigma 0}$, $w \in \mathbb{H}^{n+1}_0$ and proceed in two steps; let us recall that $v{\bullet}\partial w$ has zero mean, see Lemma 2.1.
Step 1. The Fourier coefficients of $v{\bullet}\partial w$, and some estimates for them. First of all $(v{\bullet}\partial w)_0 = 0$. The other Fourier coefficients are

$$(v{\bullet}\partial w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h\in\mathbf{Z}^d_{0k}} [v_h{\bullet}\,(k-h)]w_{k-h} \qquad \text{for } k \in \mathbf{Z}^d_0 \ ; \tag{4.9}$$

this follows from (2.28) taking into account that, in the sum therein, the term with $h = 0$ vanishes due to $v_0 = 0$, and the term with $h = k$ is zero for evident reasons.
Taking (4.9) as a starting point, let us make some remarks on the term $v_h{\bullet}\,(k-h)$ appearing therein. We have $h{\bullet}v_h = 0$ due to the assumption $\operatorname{div} v = 0$; so, we can apply Eq. (4.2) with $p = h$, $q = k - h$ and $z = v_h$, which gives

$$|v_h{\bullet}\,(k-h)| \leqslant \frac{|h \wedge (k-h)|}{|h|}|v_h| = \frac{|h\wedge k|}{|h|}|v_h| \tag{4.10}$$

(recall that $h \wedge (k-h) = h \wedge k$).
Eqs. (4.9) and (4.10) imply the following, for each $k \in \mathbf{Z}^d_0$:

$$|(v{\bullet}\partial w)_k| \leqslant \frac{1}{(2\pi)^{d/2}} \sum_{h\in\mathbf{Z}^d_{0k}} \frac{|h\wedge k|}{|h|}|v_h||w_{k-h}|$$

$$= \frac{1}{(2\pi)^{d/2}} \sum_{h\in\mathbf{Z}^d_{0k}} \frac{|h\wedge k|}{|h|^{n+1}|k-h|^{n+1}} \Big(|h|^n|v_h||k-h|^{n+1}|w_{k-h}|\Big) \ ; \tag{4.11}$$

now, Hölder's inequality $|\sum_h \ a_h b_h|^2 \leqslant \Big(\sum_h |a_h|^2\Big)\Big(\sum_h \ |b_h|^2\Big)$ gives

$$|(v{\bullet}\partial w)_k|^2 \leqslant \frac{1}{(2\pi)^d}\,\mathcal{C}_n(k)\mathcal{D}_n(k) \text{ for all } k \in \mathbf{Z}^d_0, \tag{4.12}$$

$$\mathcal{C}_n(k) := \sum_{h\in\mathbf{Z}^d_{0k}} \frac{|h\wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} \ ,$$

$$\mathcal{D}_n(k) \equiv \mathcal{D}_n(v,w)(k) := \sum_{h\in\mathbf{Z}^d_{0k}} |h|^{2n}|v_h|^2|k-h|^{2n+2}|w_{k-h}|^2$$

(in the definition of $\mathcal{D}_n(k)$ one can write as well $\sum_{h\in\mathbf{Z}^d_0}$, since the general term of the

sum vanishes for $h = k$). We now multiply both sides of (4.12) by $|k|^{2n}$; it appears that $|k|^{2n}\mathcal{C}_n(k) = \mathcal{K}_n(k)$ with $\mathcal{K}_n(k)$ as in (3.13), so

$$|k|^{2n}|(v{\bullet}\partial w)_k|^2 \leqslant \frac{1}{(2\pi)^d}\mathcal{K}_n(k)\mathcal{D}_n(k) \ . \tag{4.13}$$

Step 2. Completing the proofs of Propositions 3.1, 3.7. Due to (4.13),

$$\sum_{k\in\mathbf{Z}^d_0} |k|^{2n}|(v{\bullet}\partial w)_k|^2 \leqslant \frac{1}{(2\pi)^d}\sum_{k\in\mathbf{Z}^d_0}\mathcal{K}_n(k)\mathcal{D}_n(k) \leqslant \frac{1}{(2\pi)^d}\Big(\sup_{k\in\mathbf{Z}^d_0}\mathcal{K}_n(k)\Big)\Big(\sum_{k\in\mathbf{Z}^d_0}\mathcal{D}_n(k)\Big) \ .$$

The sup of $\mathcal{K}_n$ is finite, as we will show (by an independent argument) in Proposition B.1. Making reference to the definition of K^+_n in terms of this sup (see Eq. (3.20)), we can write the last result as

$$\sum_{k\in\mathbf{Z}^d_0} |k|^{2n}|(v{\bullet}\partial w)_k|^2 \leqslant (K^+_n)^2 \sum_{k\in\mathbf{Z}^d_0}\mathcal{D}_n(k) \ . \tag{4.14}$$

On the other hand, making explicit the definition of $\mathcal{D}_n$ we see that

$$\sum_{k\in\mathbf{Z}^d_0}\mathcal{D}_n(k) = \sum_{k\in\mathbf{Z}^d_0}\sum_{h\in\mathbf{Z}^d_0}|h|^{2n}|v_h|^2|k-h|^{2n+2}|w_{k-h}|^2 \tag{4.15}$$

$$= \sum_{h\in\mathbf{Z}^d_0}|h|^{2n}|v_h|^2\sum_{k\in\mathbf{Z}^d_0}|k-h|^{2(n+1)}|w_{k-h}|^2 = \Big(\sum_{h\in\mathbf{Z}^d_0}|h|^{2n}|v_h|^2\Big)\Big(\sum_{\ell\in\mathbf{Z}^d_{0h}}|\ell|^{2(n+1)}|w_\ell|^2\Big)$$

$$\leqslant \Big(\sum_{h\in\mathbf{Z}^d_0}|h|^{2n}|v_h|^2\Big)\ \Big(\sum_{\ell\in\mathbf{Z}^d_0}|\ell|^{2(n+1)}|w_\ell|^2\Big) = \|v\|^2_n\,\|w\|^2_{n+1} \ .$$

Returning to (4.14), we obtain

$$\sum_{k\in\mathbf{Z}^d_0} |k|^{2n}|(v{\bullet}\partial w)_k|^2 \leqslant (K^+_n)^2\|v\|^2_n\,\|w\|^2_{n+1} \ . \tag{4.16}$$

We already know that $v{\bullet}\partial w$ has zero mean. Eq. (4.16) indicates the finiteness of $\sum_{k\in\mathbf{Z}^d_0}|k|^{2n}|(v{\bullet}\partial w)_k|^2$, so

$$v{\bullet}\partial w \in \mathbb{H}^n_0 \ ; \tag{4.17}$$

Eq. (4.16) also gives

$$\|v{\bullet}\partial w\|_n \leqslant K^+_n\|v\|_n\|w\|_{n+1} \ . \tag{4.18}$$

Now, we let (v,w) vary. The map $\mathbb{H}^n_{\Sigma 0}\times\mathbb{H}^{n+1}_0 \to \mathbb{H}^n_0$, $(v,w)\mapsto v{\bullet}\partial w$ is clearly bilinear, and (4.18) indicates its continuity; so, Proposition 3.1 is proved.
Eq. (4.18) indicates as well that the sharp constant K'_n in the inequality $\|v{\bullet}\partial w\|_n \leqslant K'_n\|v\|_n\|w\|_{n+1}$ fulfills $K'_n \leqslant K^+_n$, thus proving Eq. (3.19) and Proposition 3.7. □

Proof of Proposition 3.8. Consider any $v \in \mathbb{H}^n_{\Sigma 0} \setminus \{0\}, w \in \mathbb{H}^{n+1}_{\Sigma 0} \setminus \{0\}$; then

$$K_n \geqslant \frac{\|\mathfrak{L}(v \bullet \partial w)\|_n}{\|v\|_n \|w\|_{n+1}} \ . \tag{4.19}$$

Hereafter we choose v, w with Fourier coefficients

$$v_k = A\delta_{k,a} + \overline{A}\delta_{k,-a} \ , \quad w_k = B\delta_{k,b} + \overline{B}\delta_{k,-b} \tag{4.20}$$

(δ the usual Kronecker symbol), where

$$a := (1, 0, ..., 0), \quad b := (0, 1, 0, ..., 0) \tag{4.21}$$

$$A, B \in \mathbf{C}^d \setminus \{0\} \ , \quad A \bullet a = 0 \ , \quad B \bullet b = 0 \ ; \tag{4.22}$$

the above conditions on A, B are fulfilled if and only if

$$A = (0, \alpha, \mathfrak{a}), \ B = (\beta, 0, \mathfrak{b}), \quad \alpha, \beta \in \mathbf{C}, \ \mathfrak{a}, \mathfrak{b} \in \mathbf{C}^{d-2}, \ (\alpha, \mathfrak{a}), (\beta, \mathfrak{b}) \neq (0, 0) \ . \tag{4.23}$$

(In the case $d = 2$, one understands $\mathfrak{a}, \mathfrak{b}$ to be missing from the above formulas: $A = (0, \alpha)$, $B = (\beta, 0)$.)
Of course, Eqs. (4.20-4.22) ensure $v, w \in \mathbb{H}^m_{\Sigma 0}$ for all $m \in \mathbf{N}$ (in particular, v, w are divergence free due to $a \bullet A = 0$, $b \bullet B = 0$). We now compute the right hand side of Eq. (4.19), in several steps.
Step 1. The norms $\|v\|_n$, $\|w\|_{n+1}$. From the Fourier representation of $\| \ \|_n$ and from (4.20), one gets $\|v\|^2_n = |a|^{2n}|A|^2 + |a|^{2n}|\overline{A}|^2$, whence (noting that $|a| = 1$)

$$\|v\|^2_n = 2|A|^2 \ = 2(|\alpha|^2 + |\mathfrak{a}|^2) \ ; \tag{4.24}$$

similarly,

$$\|w\|^2_{n+1} = 2|B|^2 = 2(|\beta|^2 + |\mathfrak{b}|^2) \ . \tag{4.25}$$

Step 2. The Fourier coefficients of $\mathfrak{L}(v \bullet \partial w)$. Let $k \in \mathbf{Z}^d_0$; from Eqs. (2.28) and (4.20-4.22) we get

$$(2\pi)^{d/2}[\mathfrak{L}(v \bullet \partial w)]_k = i \sum_{h = \pm a} [v_h \bullet (k - h)]\mathfrak{L}_k w_{k-h} \tag{4.26}$$

$$= i[A \bullet (k - a)]\mathfrak{L}_k w_{k-a} + i[\overline{A} \bullet (k + a)]\mathfrak{L}_k w_{k+a} = i(A \bullet k)\mathfrak{L}_k w_{k-a} + i(\overline{A} \bullet k)\mathfrak{L}_k w_{k+a}$$
$$= i(A \bullet k)(\delta_{k-a,b}\mathfrak{L}_k B + \delta_{k-a,-b}\mathfrak{L}_k \overline{B}) + i(\overline{A} \bullet k)(\delta_{k+a,b}\mathfrak{L}_k B + \delta_{k+a,-b}\mathfrak{L}_k \overline{B}) \ .$$

On the other hand,

$$(A\bullet k)\delta_{k-a,b}\mathfrak{L}_k B = (A\bullet b)\delta_{k,a+b}\mathfrak{L}_{a+b}B \ ,$$
$$(A\bullet k)\delta_{k-a,-b}\mathfrak{L}_k \overline{B} = -(A\bullet b)\delta_{k,a-b}\mathfrak{L}_{a-b}\overline{B} \ ,$$
$$(\overline{A}\bullet k)\delta_{k+a,b}\mathfrak{L}_k B = (\overline{A}\bullet b)\delta_{k,-a+b}\mathfrak{L}_{-a+b}B = (\overline{A}\bullet b)\delta_{k,-a+b}\mathfrak{L}_{a-b}B \ ,$$
$$(\overline{A}\bullet k)\delta_{k+a,-b}\mathfrak{L}_k \overline{B} = -(\overline{A}\bullet b)\delta_{k,-a-b}\mathfrak{L}_{-a-b}\overline{B} = -(\overline{A}\bullet b)\delta_{k,-a-b}\mathfrak{L}_{a+b}\overline{B} \ ;$$

so, returning to (4.26) we get

$$(2\pi)^{d/2}[\mathfrak{L}(v\bullet\partial w)]_k \tag{4.27}$$
$$= i(A\bullet b)(\delta_{k,a+b}\mathfrak{L}_{a+b}B - \delta_{k,a-b}\mathfrak{L}_{a-b}\overline{B}) - i(\overline{A}\bullet b)(\delta_{k,-a-b}\mathfrak{L}_{a+b}\overline{B} - \delta_{k,-a+b}\mathfrak{L}_{a-b}B) \ .$$

The explicit expressions (4.21) (4.22) for a, b, A, B give

$$A\bullet b = \alpha, \quad \overline{A}\bullet b = \overline{\alpha} \ ; \tag{4.28}$$

$$\mathfrak{L}_{a\pm b}B = B - \frac{(a\pm b)\bullet B}{|a\pm b|}(a\pm b) = (\beta, 0, \mathfrak{b}) - \frac{\beta}{\sqrt{2}}(1, \pm 1, 0, ..., 0) \tag{4.29}$$
$$= \left(\left(1-\frac{1}{\sqrt{2}}\right)\beta, \mp\frac{\beta}{\sqrt{2}}, \mathfrak{b}\right) \ ;$$
$$\mathfrak{L}_{a\pm b}\overline{B} = \overline{\mathfrak{L}_{a\pm b}B} = \left(\left(1-\frac{1}{\sqrt{2}}\right)\overline{\beta}, \mp\frac{\overline{\beta}}{\sqrt{2}}, \overline{\mathfrak{b}}\right) \ . \tag{4.30}$$

Inserting Eqs. (4.28-4.30) into (4.27), we obtain the final result

$$[\mathfrak{L}(v\bullet\partial w)]_k = \frac{i\alpha}{(2\pi)^{d/2}}\left[\left(\left(1-\frac{1}{\sqrt{2}}\right)\beta, -\frac{\beta}{\sqrt{2}}, \mathfrak{b}\right)\delta_{k,a+b} - \left(\left(1-\frac{1}{\sqrt{2}}\right)\overline{\beta}, \frac{\overline{\beta}}{\sqrt{2}}, \overline{\mathfrak{b}}\right)\delta_{k,a-b}\right]$$
$$-\frac{i\overline{\alpha}}{(2\pi)^{d/2}}\left[\left(\left(1-\frac{1}{\sqrt{2}}\right)\overline{\beta}, -\frac{\overline{\beta}}{\sqrt{2}}, \overline{\mathfrak{b}}\right)\delta_{k,-a-b} - \left(\left(1-\frac{1}{\sqrt{2}}\right)\beta, \frac{\beta}{\sqrt{2}}, \mathfrak{b}\right)\delta_{k,-a+b}\right]. \tag{4.31}$$

Step 3. The norm of $\mathfrak{L}(v\bullet\partial w)$. From (4.31) and the Fourier representation of $\| \ \|_n$ we get

$$\|\mathfrak{L}(v\bullet\partial w)\|_n^2 = \sum_{k=a\pm b, -a\mp b} |k|^{2n} |[\mathfrak{L}(v\bullet\partial w)]_k|^2 \tag{4.32}$$
$$= \frac{2^n}{(2\pi)^d} \, 4|\alpha|^2 \left(\left(1-\frac{1}{\sqrt{2}}\right)^2 |\beta|^2 + \frac{1}{2}|\beta|^2 + |\mathfrak{b}|^2\right) = \frac{2^{n+2}}{(2\pi)^d}|\alpha|^2\left((2-\sqrt{2})|\beta|^2 + |\mathfrak{b}|^2\right) \ .$$

Step 4. The lower bound on K_n. We return to the inequality (4.19), using the expressions (4.24), (4.25), (4.32) for the norms of $v, w, \mathfrak{L}(v\bullet\partial w)$; this gives

$$K_n^2 \geqslant \frac{2^n}{(2\pi)^d}\frac{|\alpha|^2\left((2-\sqrt{2})|\beta|^2 + |\mathfrak{b}|^2\right)}{(|\alpha|^2+|\mathfrak{a}|^2)(|\beta|^2+|\mathfrak{b}|^2)} \tag{4.33}$$

for all $(\alpha, \mathfrak{a}), (\beta, \mathfrak{b}) \in \mathbf{C} \times \mathbf{C}^{d-2} \setminus \{(0,0)\}$.
In the case $d = 2$, one understands $\mathfrak{a}, \mathfrak{b}$ to be missing from the above formula; so, (4.33) gives

$$K_n^2 \geqslant \frac{2^n}{(2\pi)^2}(2 - \sqrt{2}) \ . \tag{4.34}$$

In the case $d \geqslant 3$, we choose $(\alpha, \mathfrak{a}), (\beta, \mathfrak{b}) \neq (0,0)$ so as to maximize the right hand side of Eq. (4.33). The maximum is attained with $\mathfrak{a} = 0$, $\beta = 0$ and arbitrary $\alpha \in \mathbf{C} \setminus \{0\}$, $\mathfrak{b} \in \mathbf{C}^{d-2} \setminus \{0\}$; this choice gives

$$K_n^2 \geqslant \frac{2^n}{(2\pi)^d} \qquad (d \geqslant 3). \tag{4.35}$$

The results (4.34) (4.35) are summarized by Eqs. (3.21-3.22) in the statement of the Proposition, which is now proved. $\square$

A Some tools preparing the analysis of the function $\mathcal{K}_n$

Let us fix some notations, to be used throughout the Appendices.

A.1 ***Definition.*** *(i) $\theta : \mathbf{R} \to \{0,1\}$ is the Heaviside function such that $\theta(z) := 1$ if $z \in [0, +\infty)$ and $\theta(z) := 0$ if $z \in (-\infty, 0)$.*
(ii) Γ is the Euler Gamma function, $\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$ are the binomial coefficients.
(iii) We put $\mathbf{S}^{d-1} := \{u \in \mathbf{R}^d \mid |u| = 1\}$. For each $p \in \mathbf{R}^d \setminus \{0\}$, the versor of p is $\widehat{p} := \dfrac{p}{|p|} \in \mathbf{S}^{d-1}$.

A.2 ***Lemma.*** *For any function $f : \mathbf{Z}_0^d \to \mathbf{R}$ and $k \in \mathbf{Z}_0^d$, $\rho \in (1, +\infty)$, one has*

$$\sum_{h \in \mathbf{Z}^d_{0k}, |h|<\rho \, or \, |k-h|<\rho} f(h) = \sum_{h \in \mathbf{Z}^d_{0k}, |h|<\rho} f(h) + \theta(|k-h| - \rho) f(k-h) \ . \tag{A.1}$$

Proof. The domain of the sum is the disjoint union of the sets $\{h \in \mathbf{Z}^d_{0k} \mid |h| < \rho\}$ and $\{h \in \mathbf{Z}^d_{0k} \mid |k-h| < \rho, |h| \geqslant \rho\}$; so,

$$\sum_{h \in \mathbf{Z}^d_0, |h|<\rho \, \mathrm{or} \, |k-h|<\rho} f(h) = \sum_{h \in \mathbf{Z}^d_{0k}, |h|<\rho} f(h) + \sum_{h \in \mathbf{Z}^d_{0k}, |k-h|<\rho, |h| \geqslant \rho} f(h)$$

$$= \sum_{h \in \mathbf{Z}^d_{0k}, |h|<\rho} f(h) + \sum_{h \in \mathbf{Z}^d_{0k}, |k-h|<\rho} \theta(|h| - \rho) \, f(h) \ .$$

Now, a change of variable $h \to k - h$ in the last sum gives the thesis (A.1). $\square$

A.3 ***Lemma.*** *For any $n \in [1, +\infty)$ and $p, q \in \mathbf{R}^d$, one has*

$$|p \wedge q|^2 |p+q|^{2n} \leqslant \frac{2^{2n+1}(n+1)^{n+1}}{(n+2)^{n+2}} \, |p|^2 |q|^2 (|p|^{2n} + |q|^{2n}) \; . \tag{A.2}$$

Proof. Eq. (A.2) is obvious if $p = 0$ or $q = 0$, due to the vanishing of both sides; hereafter we prove (A.2) for $p, q \in \mathbf{R}^d \setminus \{0\}$. Let $\vartheta \in [0, \pi]$ denote the convex angle between p and q; then

$$|p \wedge q|^2 = |p|^2 |q|^2 (1 - \cos^2 \vartheta) \; , \quad |p+q|^2 = |p|^2 + |q|^2 + 2|p||q| \cos \vartheta \; ,$$

so

$$\frac{|p \wedge q|^2 |p+q|^{2n}}{|p|^2 |q|^2 (|p|^{2n} + |q|^{2n})} \tag{A.3}$$

$$= \frac{(1 - \cos^2 \vartheta)(|p|^2 + |q|^2 + 2|p||q| \cos \vartheta)^n}{|p|^{2n} + |q|^{2n}} = b_n\Big(\cos \vartheta, \frac{|p|}{|q|} \Big),$$

having put

$$b_n : [-1, 1] \times [0, +\infty) \to [0, +\infty) \; , \tag{A.4}$$

$$(c, \xi) \mapsto b_n(c, \xi) := \frac{(1 - c^2)(1 + 2c\xi + \xi^2)^n}{1 + \xi^{2n}} \; .$$

One checks by elementary tools that

$$\sup_{c \in [-1,1], \, \xi \in [0, +\infty)} b_n(c, \xi) = b_n\Big(\frac{n}{n+2}, 1 \Big) = \frac{2^{2n+1}(n+1)^{n+1}}{(n+2)^{n+2}}. \tag{A.5}$$

Returning to Eq. (A.3), and writing $b_n\Big(\cos \vartheta, \dfrac{|p|}{|q|} \Big) \leqslant \dfrac{2^{2n+1}(n+1)^{n+1}}{(n+2)^{n+2}}$ we obtain the thesis (A.2). $\square$

A.4 ***Lemma.*** *Let $\nu \in (d, +\infty)$. For any $\rho \in (2\sqrt{d}, +\infty)$, one has*

$$\sum_{h \in \mathbf{Z}^d, |h| \geqslant \rho} \frac{1}{|h|^\nu} \leqslant \frac{2\pi^{d/2}}{\Gamma(d/2)} \sum_{i=0}^{d-1} \begin{pmatrix} d-1 \\ i \end{pmatrix} \frac{d^{d/2 - 1/2 - i/2}}{(\nu - 1 - i)(\rho - 2\sqrt{d})^{\nu - 1 - i}} \; . \tag{A.6}$$

Proof. This is just Lemma C.2 of [8] (with the variable λ of the cited reference related to ρ by $\lambda = \rho - 2\sqrt{d}$). $\square$

A.5 *Lemma.* *Let $\rho \in (1, +\infty)$ and $\varphi : [1, \rho) \to \mathbf{R}$. Then, for each $k \in \mathbf{R}^d$,*

$$\sum_{h \in \mathbf{Z}^d_0, |h| < \rho} (h \bullet k)^2 \varphi(|h|) = \frac{|k|^2}{d} \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} |h|^2 \varphi(|h|) \ . \tag{A.7}$$

Proof. We reexpress the left hand side of (A.7) writing $(h \bullet k)^2 = (\sum_{r=1}^d h_r k_r) \times (\sum_{s=1}^d h_s k_s) = \sum_{r,s=1}^d h_r h_s k_r k_s$, which gives

$$\sum_{h \in \mathbf{Z}^d_0, |h| < \rho} (h \bullet k)^2 \varphi(|h|) = \sum_{r,s=1}^d \mathcal{Y}_{rs} k_r k_s \qquad \mathcal{Y}_{rs} := \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} h_r h_s \varphi(|h|) \ . \tag{A.8}$$

From the definition of $\mathcal{Y}_{rs}$, one easily checks that

$$\mathcal{Y}_{rs} = 0 \text{ for } r \neq s, \quad \mathcal{Y}_{11} = \mathcal{Y}_{22} = ... = \mathcal{Y}_{dd} \ . \tag{A.9}$$

By the second of the above statements, for each $r \in \{1, ..., d\}$ we have

$$\mathcal{Y}_{rr} = \frac{1}{d} \sum_{s=1}^d \mathcal{Y}_{ss} = \frac{1}{d} \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} \sum_{s=1}^d h_s^2 \varphi(|h|) = \frac{1}{d} \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} |h|^2 \varphi(|h|) \ ;$$

in conclusion,

$$\mathcal{Y}_{rs} = \frac{\delta_{rs}}{d} \sum_{h \in \mathbf{Z}^d_0, |h| < \rho} |h|^2 \varphi(|h|) \qquad (r, s = 1, ..., d) \ . \tag{A.10}$$

Inserting this result into the first equality (A.8), we obtain the thesis (A.7). $\square$

A.6 *Definition.* *Let us introduce the domain*

$$\mathcal{E} := \{(c, \xi) \in \mathbf{R}^2 \mid c \in [-1, 1], \, \xi \in [0, +\infty), \, (c, \xi) \neq (1, 1)\} \ ; \tag{A.11}$$

furthermore, let $n \in \mathbf{R}$.
(i) E_n is the C^∞ function defined as follows:

$$E_n : \mathcal{E} \to [0, +\infty) \ , \quad (c, \xi) \mapsto E_n(c, \xi) := \frac{1 - c^2}{(1 - 2c\xi + \xi^2)^{n+1}} \ . \tag{A.12}$$

(ii) For $\ell = 0, 1, 2, ...,$ we put

$$E_{n\ell} : [-1, 1] \to \mathbf{R} \ , \qquad c \mapsto E_{n\ell}(c) := \frac{1}{\ell!} \frac{\partial^\ell E_n}{\partial \xi^\ell}(c, 0) \ . \tag{A.13}$$

(iii) For $t = 1, 2, ...,$

$$R_{nt} : \mathcal{E} \to \mathbf{R} \ , \tag{A.14}$$

is the unique C^∞ *function such that, for all* $(c, \xi) \in \mathcal{E}$*,*

$$E_n(c, \xi) = \sum_{\ell=0}^{t-1} E_{n\ell}(c)\xi^\ell + R_{nt}(c, \xi)\xi^t \ . \tag{A.15}$$

(iv) For $t = 1, 2, ...,$ *we put*

$$\mu_{nt} := \min_{c\in[-1,1],\,\xi\in[0,1/2]} R_{nt}(c, \xi) \ , \qquad M_{nt} := \max_{c\in[-1,1],\,\xi\in[0,1/2]} R_{nt}(c, \xi) \ . \tag{A.16}$$

A.7 Remarks. (i) Of course, E_n could be defined on a domain larger than $\mathcal{E}$; this is not relevant for our purposes.
(ii) Some calculations give

$$E_{n0}(c) = 1 - c^2 \ , \qquad E_{n1}(c) = (2n + 2)(c - c^3) \ , \tag{A.17}$$

$$E_{n2}(c) = -(n + 1) + (2n^2 + 7n + 5)c^2 - (2n^2 + 6n + 4)c^4 \ ,$$

etc.
(iii) In general, $E_{n\ell}$ is a polynomial in c of degree $\ell + 2$; this polynomial is *even* ($E_{n\ell}(-c) = E_{n\ell}(c)$) for even ℓ, and *odd* ($E_{n\ell}(-c) = -E_{n\ell}(c)$) for odd ℓ.
(iv) Eq. (A.15) indicates that $R_{nt}(c, \xi)\xi^t$ is the reminder in the Taylor expansion of $E_n(c, \xi)$ at order t in ξ, about the point $\xi = 0$. For the practical computation of R_{nt} one can note that Eq. (A.15) (and Taylor's formula) imply

$$R_{nt}(c, \xi) = \begin{cases} \xi^{-t}\left(E_n(c, \xi) - \sum_{\ell=0}^{t-1} E_{n\ell}(c)\xi^\ell\right) & \text{if } \xi \neq 0, \\ E_{nt}(c) & \text{if } \xi = 0. \end{cases} \tag{A.18}$$

(v) The minimum μ_{nt} and the maximum M_{nt} in (A.16) exist, since we consider the continuous function R_{nt} on a compact domain. For specific values of n and t, these can be evaluated numerically starting from the representation (A.18) of R_{nt}. In this way we obtain, for example, the values

$$\mu_{26} = -22.720... \, , \ M_{26} = 73.835... \, ; \ \ \mu_{36} = -61.239... \, , \ M_{36} = 410.74... \, ; \tag{A.19}$$

$$\mu_{46} = -135.89... \, , \ M_{46} = 1832.8... \, ; \ \ \mu_{56} = -264.44... \, , \ M_{56} = 7252.9... \, ;$$

$$\mu_{10,6} = -2582.5... \, , \ M_{10,6} = 4.6371... \times 10^6 \, ,$$

recorded here for subsequent use.
(vi) For arbitrary n and t, Eqs. (A.15) (A.16) imply

$$\sum_{\ell=0}^{t-1} E_{n\ell}(c)\xi^\ell + \mu_{nt}\xi^t \leqslant E_n(c, \xi) \leqslant \sum_{\ell=0}^{t-1} E_{n\ell}(c)\xi^\ell + M_{nt}\xi^t \tag{A.20}$$

$$\text{for } (c, \xi) \in [-1, 1] \times [0, 1/2]. \qquad \square$$

Hereafter we present a lemma about the function $k, h \mapsto \dfrac{|k|^{2n}|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}}$, appearing in the definition (3.13) of $\mathcal{K}_n$; as indicated by the Lemma, this is related to the function E_n of Definition A.6 and to its Taylor expansion.

A.8 ***Lemma.*** *Let $h, k \in \mathbf{R}^d \setminus \{0\}$, $h \neq k$, and let $\vartheta(h,k) \equiv \vartheta$ be the convex angle between them. Furthermore, let $n \in \mathbf{R}$; then the following holds.*
(i) One has

$$\frac{|k|^{2n}|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} = \frac{1}{|h|^{2n}} E_n\Big(\cos\vartheta, \frac{|h|}{|k|}\Big) \ . \tag{A.21}$$

(ii) Let $|k| \geqslant 2|h|$. For $t \in \{1,2,...\}$, Eq. (A.21) implies

$$\sum_{\ell=0}^{t-1} \frac{E_{n\ell}(\cos\vartheta)}{|h|^{2n-\ell}|k|^{\ell}} + \frac{m_{nt}}{|h|^{2n-t}|k|^t} \leqslant \frac{|k|^{2n}|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} \tag{A.22}$$

$$\leqslant \sum_{\ell=0}^{t-1} \frac{E_{n\ell}(\cos\vartheta)}{|h|^{2n-\ell}|k|^{\ell}} + \frac{M_{nt}}{|h|^{2n-t}|k|^t}$$

(note that $\cos\vartheta = \widehat{h} \bullet \widehat{k}$).

Proof. (i) Writing $|h \wedge k|^2 = |h|^2|k|^2(1-\cos^2\vartheta)$ and $|k-h|^2 = |k|^2 - 2|k||h|\cos\vartheta$ $+|h|^2$, we readily obtain

$$\frac{|k|^{2n}|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} = \frac{1}{|h|^{2n}} \frac{1-\cos^2\vartheta}{(1-2\cos\vartheta|h|/|k| + |h|^2/|k|^2)^{n+1}} \ ;$$

comparing this with the definition (A.12) of E_n, we obtain the thesis (A.21).
(ii) It suffices to use Eq. (A.21) and the inequalities (A.20), with $c := \cos\vartheta$ and $\xi := |h|/|k|$; note that $0 \leqslant \xi \leqslant 1/2$ due to the assumption $|k| \geqslant 2|h|$. $\square$

To conclude, we introduce some variants of the polynomials $E_{n\ell}$, to be used in the sequel.

A.9 ***Definition.*** *For $\ell = 0, 2, ...,$, $\hat{E}_{n\ell d} \equiv \hat{E}_{n\ell}$ are the polynomials $E_{n\ell}$ of Eq. (A.13), where the term c^2 has been replaced with $1/d$.*

A.10 Example. The expressions of E_{n0}, E_{n2} in (A.17) imply

$$\hat{E}_{n0}(c) = \text{const.} = 1 - \frac{1}{d} \ , \tag{A.23}$$

$$\hat{E}_{n2}(c) = -(n+1) + \frac{2n^2+7n+5}{d} - (2n^2+6n+4)c^4 \ .$$

B The function $\mathcal{K}_n$

Throughout the Appendix, $n \in (d/2, +\infty)$. For $k \in \mathbf{Z}^d_0$, let us recall the definition (3.13)

$$\mathcal{K}_n(k) := |k|^{2n} \sum_{h \in \mathbf{Z}^d_{0k}} \frac{|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} \in (0, +\infty) \ .$$

B.1 ***Proposition.*** *Let us fix a "cutoff"*

$$\rho \in (2\sqrt{d}, +\infty) \ ; \tag{B.1}$$

then, the following holds (with the functions and quantities $\mathcal{K}_n$, $\delta\mathcal{K}_n$,... mentioned in the sequel depending parametrically on d and ρ: $\mathcal{K}_n(k) \equiv \mathcal{K}_{nd}(\rho, k)$, $\delta\mathcal{K}_n \equiv \delta\mathcal{K}_{nd}(\rho)$,...).
(i) The function $\mathcal{K}_n$ can be evaluated using the inequalities

$$\mathcal{K}_n(k) < \mathcal{K}_n(k) \leqslant \mathcal{K}_n(k) + \delta\mathcal{K}_n \quad \textit{for all } k \in \mathbf{Z}^d_0 \ . \tag{B.2}$$

Here

$$\mathcal{K}_n(k) := |k|^{2n} \sum_{h \in \mathbf{Z}^d_{0k}, |h|<\rho \, or \, |k-h|<\rho} \frac{|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} \ ; \tag{B.3}$$

this can be reexpressed as

$$\mathcal{K}_n(k) = |k|^{2n} \sum_{h \in \mathbf{Z}^d_{0k}, |h|<\rho} [1 + \theta(|k-h| - \rho)] \frac{|h \wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} \tag{B.4}$$

(with θ as in Definition A.1). If $|k| \geqslant 2\rho$, in Eq. (B.4) one can replace $\mathbf{Z}^d_{0k}$ with $\mathbf{Z}^d_0$ and $\theta(|k-h| - \rho)$ with 1.
Finally,

$$\delta\mathcal{K}_n := \frac{2^{2n+3}\pi^{d/2}(n+1)^{n+1}}{\Gamma(d/2)(n+2)^{n+2}} \sum_{i=0}^{d-1} \begin{pmatrix} d-1 \\ i \end{pmatrix} \frac{d^{d/2-1/2-i/2}}{(2n-i-1)(\rho - 2\sqrt{d})^{2n-i-1}} \ . \tag{B.5}$$

(ii) As in Eq. (3.16), consider the reflection operators R_r ($r = 1, ..., d$) and the permutation operators P_σ (σ a permutation of $\{1, ..., d\}$). Then

$$\mathcal{K}_n(R_r k) = \mathcal{K}_n(k) \ , \quad \mathcal{K}_n(P_\sigma k) = \mathcal{K}_n(k) \qquad \textit{for each } k \in \mathbf{Z}^d_0 \tag{B.6}$$

(so, the computation of $\mathcal{K}_n(k)$ can be reduced to the case $k_1 \geqslant k_2 \geqslant ... \geqslant k_d \geqslant 0$).

(iii) Let $t \in \{2,4,6,...\}$. One has

$$Z_n + \sum_{\ell=2,4,....,t-2} \frac{\mathcal{Q}_{n\ell}(\widehat{k})}{|k|^\ell} + \frac{v_{nt}}{|k|^t} \leqslant \mathcal{K}_n(k) \leqslant Z_n + \sum_{\ell=2,4,....,t-2} \frac{\mathcal{Q}_{n\ell}(\widehat{k})}{|k|^\ell} + \frac{V_{nt}}{|k|^t}$$

$$\textit{for } k \in \mathbf{Z}^d_0,\ |k| \geqslant 2\rho. \tag{B.7}$$

In the above $\sum_{\ell=2,...,t-2} \cdots := 0$ if $t = 2$, and $\widehat{k} \in \mathbf{S}^{d-1}$ is the versor of k (see Definition A.1); furthermore,

$$Z_n := 2\left(1 - \frac{1}{d}\right) \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{1}{|h|^{2n}}\ ; \tag{B.8}$$

$$v_{nt} := 2m_{nt} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{1}{|h|^{2n-t}}, \quad V_{nt} := 2M_{nt} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{1}{|h|^{2n-t}} \quad (m_{nt}, M_{nt} \textit{ as in (A.16)})\ ; \tag{B.9}$$

$$\mathcal{Q}_{n\ell} : \mathbf{S}^{d-1} \to \mathbf{R},\ u \mapsto \mathcal{Q}_{n\ell}(u) := 2 \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{\hat{E}_{n\ell}(u \bullet \widehat{h})}{|h|^{2n-\ell}}\ (\hat{E}_{n\ell} \textit{ as in Definition A.9}). \tag{B.10}$$

For each ℓ, $\mathcal{Q}_{n\ell}$ is a polynomial function on $\mathbf{S}^{d-1}$; setting

$$q_{n\ell} := \min_{u \in \mathbf{S}^{d-1}} \mathcal{Q}_{n\ell}(u)\ , \qquad Q_{n\ell} := \max_{u \in \mathbf{S}^{d-1}} \mathcal{Q}_{n\ell}(u)\ , \tag{B.11}$$

one infers from (B.7) that

$$Z_n + \sum_{\ell=2,4,....,t-2} \frac{q_{n\ell}}{|k|^\ell} + \frac{v_{nt}}{|k|^t} \leqslant \mathcal{K}_n(k) \leqslant Z_n + \sum_{\ell=2,4,....,t-2} \frac{Q_{n\ell}}{|k|^\ell} + \frac{V_{nt}}{|k|^t}$$

$$\textit{for } k \in \mathbf{Z}^d_0,\ |k| \geqslant 2\rho\ . \tag{B.12}$$

These facts imply

$$\mathcal{K}_n(k) \to Z_n \qquad \textit{for } k \to \infty\ . \tag{B.13}$$

(iv) Items (i) and (iii) imply

$$\sup_{k \in \mathbf{Z}^d_0} \mathcal{K}_n(k) \leqslant \sup_{k \in \mathbf{Z}^d_0} \mathcal{K}_n(k) \leqslant \Big(\sup_{k \in \mathbf{Z}^d_0} \mathcal{K}_n(k)\Big) + \delta\mathcal{K}_n < +\infty\ . \tag{B.14}$$

Proof. We fix a cutoff ρ as in (B.1), and proceed in several steps. More precisely Steps 1-5 give proofs of statements (i)(ii), while Steps 6-8 prove statements (iii)(iv). The assumption (B.1) $\rho > 2\sqrt{d}$ is essential in Step 3.

Step 1. One has

$$\mathcal{K}_n(k) = \mathcal{K}_n(k) + \Delta\mathcal{K}_n(k) \qquad \textit{for all } k \in \mathbf{Z}^d_0\ , \tag{B.15}$$

with $\mathcal{K}_n(k) := |k|^{2n} \sum_{h\in \mathbf{Z}^d_{0k}, |h|<\rho \,\mathrm{or}\, |k-h|<\rho} \frac{|h\wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}}$, *as in (B.3), and*

$$\Delta\mathcal{K}_n(k) := |k|^{2n} \sum_{h\in \mathbf{Z}^d_{0}, |h|\geqslant\rho, |k-h|\geqslant\rho} \frac{|h\wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} \in (0,+\infty) \ . \tag{B.16}$$

The above decomposition follows noting that $\mathbf{Z}^d_{0k}$ is the disjoint union of the domains of the sums defining $\mathcal{K}_n(k)$ and $\Delta\mathcal{K}_n(k)$. $\mathcal{K}_n(k)$ is finite, involving finitely many summands; $\Delta\mathcal{K}_n(k)$ is finite as well, since we know that $\mathcal{K}_n(k) < +\infty$.
Step 2. For each $k \in \mathbf{Z}^d_0$, *one has the representation (B.4)*

$$\mathcal{K}_n(k) = |k|^{2n} \sum_{h\in \mathbf{Z}^d_{0k}, |h|<\rho} [1+\theta(|k-h|-\rho)] \frac{|h\wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} \ .$$

If $|k| \geqslant 2\rho$, *in Eq.(B.4) one can replace* $\mathbf{Z}^d_{0k}$ *with* $\mathbf{Z}^d_0$ *and* $\theta(|k-h|-\rho)$ *with* 1.
To prove (B.4) we start from the definition (B.3) of $\mathcal{K}_n(k)$, and reexpress the sum therein using Eq. (A.1), with $f(h) \equiv f_k(h) := \frac{|h\wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}}$. Noting that $f(k-h) = f(h)$, we can write

$$f(h) + \theta(|k-h|-\rho) f(k-h) = [1+\theta(|k-h|-\rho)] \frac{|h\wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} \ ,$$

and obtain Eq. (B.4). Now, assume $|k| \geqslant 2\rho$; then, for all $h \in \mathbf{Z}^d_0$ with $|h| < \rho$ one has

$$|k-h| \geqslant |k| - |h| > \rho \ ; \tag{B.17}$$

this implies $h \neq k$ (i.e., $h \in \mathbf{Z}^d_{0k}$) and $\theta(|k-h|-\rho) = 1$, two facts which justify the replacements indicated above in (B.4).
Step 3. For each $k \in \mathbf{Z}^d_0$, *one has*

$$0 < \Delta\mathcal{K}_n(k) \leqslant \delta\mathcal{K}_n \ , \tag{B.18}$$

with $\delta\mathcal{K}_n$ *as in Eq. (B.5).* The obvious relation $0 < \Delta\mathcal{K}_n(k)$ was already noted; in the sequel we prove that $\Delta\mathcal{K}_n(k) \leqslant \delta\mathcal{K}_n$. To show this, we note the following: for each h in the sum (B.16), one can write

$$\begin{aligned} |h\wedge k|^2|k|^{2n} &= |h\wedge(k-h)|^2|h+(k-h)|^{2n} \\ &\leqslant B_n(|h|^{2n+2}|k-h|^2 + |h|^2|k-h|^{2n+2}) \ , \end{aligned} \tag{B.19}$$

where the last passage depends on the inequality (A.2), applied with $p = h$ and $q = k-h$; for the sake of brevity, we have put

$$B_n := \frac{2^{2n+1}(n+1)^{n+1}}{(n+2)^{n+2}} \ . \tag{B.20}$$

Eqs. (B.16) (B.19) give

$$\Delta\mathcal{K}_n(k) \leqslant B_n \Big(\sum_{h\in\mathbf{Z}_0^d, |h|\geqslant\rho, |k-h|\geqslant\rho} \frac{1}{|k-h|^{2n}} + \sum_{h\in\mathbf{Z}_0^d, |h|\geqslant\rho, |k-h|\geqslant\rho} \frac{1}{|h|^{2n}} \Big) . \tag{B.21}$$

The domain of the above two sums is contained in each one of the sets $\{h \in \mathbf{Z}_0^d \mid |k-h| \geqslant \rho\}$ and $\{h \in \mathbf{Z}_0^d \mid |h| \geqslant \rho\}$; so,

$$\Delta\mathcal{K}_n(k) \leqslant B_n \Big(\sum_{h\in\mathbf{Z}_0^d, |k-h|\geqslant\rho} \frac{1}{|k-h|^{2n}} + \sum_{h\in\mathbf{Z}_0^d, |h|\geqslant\rho} \frac{1}{|h|^{2n}} \Big) .$$

Now, the change of variable $h \mapsto k-h$ in the first sum shows that it is equal to the second one, so

$$\Delta\mathcal{K}_n(k) \leqslant 2B_n \sum_{h\in\mathbf{Z}_0^d, |h|\geqslant\rho} \frac{1}{|h|^{2n}} . \tag{B.22}$$

Finally, Eq. (B.22) and Eq. (A.6) with $\nu = 2n$ give

$$\Delta\mathcal{K}_n(k) \leqslant \frac{4\pi^{d/2} B_n}{\Gamma(d/2)} \sum_{i=0}^{d-1} \left(\begin{array}{c} d-1 \\ i \end{array} \right) \frac{d^{d/2-1/2-i/2}}{(2n-i-1)(\rho-2\sqrt{d})^{2n-i-1}} ;$$

the right hand side of this inequality is $\delta\mathcal{K}_n$ defined by (B.5), as seen immediately using the definition (B.20) of B_n.
Step 4. One has the equalities (B.6) $\mathcal{K}_n(R_r k) = \mathcal{K}_n(k)$, $\mathcal{K}_n(P_\sigma k) = \mathcal{K}_n(k)$, *involving the reflection and permutation operators* R_r, P_σ*.* The proof starts from the definition (B.3) of $\mathcal{K}_n$, and is very similar to the one employed for the analogous properties of $\mathcal{K}_n$ (see Eq. (3.17) and the subsequent comments).
Step 5. One has the inequalities (B.2) $\mathcal{K}_n(k) < \mathcal{K}_n(k) \leqslant \mathcal{K}_n(k) + \delta\mathcal{K}_n$*.* These relations follow immediately from the decomposition (B.15) $\mathcal{K}_n(k) = \mathcal{K}_n(k) + \Delta\mathcal{K}_n(k)$ and from the bounds (B.18) on $\Delta\mathcal{K}_n(k)$.
Step 6. Let $t \in \{2, 4, 6, ...\}$*; one has the inequalities (B.7) for* $\mathcal{K}_n(k)$*.* As an example, we prove the upper bound (B.7)

$$\mathcal{K}_n(k) \leqslant Z_n + \sum_{\ell=2,4,\dots,t-2} \frac{\mathcal{Q}_{n\ell}(\widehat{k})}{|k|^\ell} + \frac{V_{nt}}{|k|^t} \qquad \text{for } k \in \mathbf{Z}_0^d,\ |k| \geqslant 2\rho .$$

For k as above we can express $\mathcal{K}_n(k)$ via Eq. (B.4), replacing therein $\mathbf{Z}_{0k}^d$ with $\mathbf{Z}_0^d$ and $\theta(|k-h|-\rho)$ with 1 (see the final statement in Step 2). So,

$$\mathcal{K}_n(k) = 2 \sum_{h\in\mathbf{Z}_0^d, |h|<\rho} \frac{|k|^{2n}|h\wedge k|^2}{|h|^{2n+2}|k-h|^{2n+2}} . \tag{B.23}$$

In this expression we insert the upper bound of Eq. (A.22), writing therein $\cos\vartheta = \widehat{h}\bullet\widehat{k}$ (note that (A.22) can be used, since $|h|/|k| < \rho/(2\rho) < 1/2$ for each h in the sum). In this way we obtain

$$\mathcal{K}_n(k) \leqslant 2 \sum_{h\in\mathbf{Z}_0^d, |h|<\rho} \left[\sum_{\ell=0}^{t-1} \frac{E_{n\ell}(\widehat{h}\bullet\widehat{k})}{|h|^{2n-\ell}|k|^\ell} + \frac{M_{nt}}{|h|^{2n-t}|k|^t} \right]$$

$$= 2\sum_{\ell=0}^{t-1} \frac{1}{|k|^\ell} \sum_{h\in\mathbf{Z}_0^d, |h|<\rho} \frac{E_{n\ell}(\widehat{h}\bullet\widehat{k})}{|h|^{2n-\ell}} + \frac{2M_{nt}}{|k|^t} \sum_{h\in\mathbf{Z}_0^d, |h|<\rho} \frac{1}{|h|^{2n-t}} ;$$

comparing with the definition (B.9), we see that the last term above is just $V_{nt}/|k|^t$. Our computation can be summarized in the equation

$$\mathcal{K}_n(k) \leqslant \sum_{\ell=0}^{t-1} \frac{\mathcal{Q}_{n\ell}(\widehat{k})}{|k|^\ell} + \frac{V_{nt}}{|k|^t} , \tag{B.24}$$

where we have provisionally put

$$\mathcal{Q}_{n\ell} : \mathbf{S}^{d-1} \to \mathbf{R} , \qquad u \mapsto \mathcal{Q}_{n\ell}(u) := 2 \sum_{h\in\mathbf{Z}_0^d, |h|<\rho} \frac{E_{n\ell}(\widehat{h}\bullet u)}{|h|^{2n-\ell}} . \tag{B.25}$$

Now, the thesis follows if we prove the following relations:

$$\mathcal{Q}_{n0}(u) = Z_n \text{ as in (B.8), for all } u \in \mathbf{S}^{d-1} ; \tag{B.26}$$

$$\mathcal{Q}_{n\ell}(u) = 0 \text{ for } \ell \in \{1, 3, ..., t-1\} \text{ and all } u \in \mathbf{S}^{d-1} ; \tag{B.27}$$

$$\mathcal{Q}_{n\ell}(u) \text{ is as in (B.10), for } \ell \in \{2, 4, ..., t-2\} \text{ and all } u \in \mathbf{S}^{d-1} . \tag{B.28}$$

To prove (B.26), we proceed as follows, for any $u \in \mathbf{S}^{d-1}$: recalling that $E_{n0}(c) = 1 - c^2$ and writing $\widehat{h} = h/|h|$ we get

$$\mathcal{Q}_{n0}(u) = 2 \sum_{h\in\mathbf{Z}_0^d, |h|<\rho} \frac{1}{|h|^{2n}} - 2 \sum_{h\in\mathbf{Z}_0^d, |h|<\rho} \frac{(h\bullet u)^2}{|h|^{2n+2}} ;$$

on the other hand, the identity (A.7) with k replaced by u and $\varphi(|h|) = 1/|h|^{2n+2}$ gives $\sum_{h\in\mathbf{Z}_0^d, |h|<\rho} (h\bullet u)^2/|h|^{2n+2} = (1/d) \sum_{h\in\mathbf{Z}_0^d, |h|<\rho} 1/|h|^{2n}$. So,

$$\mathcal{Q}_{n0}(u) = 2\left(1 - \frac{1}{d}\right) \sum_{h\in\mathbf{Z}_0^d, |h|<\rho} \frac{1}{|h|^{2n}} = Z_n ,$$

and (B.26) is proved.

Let us pass to (B.27). This relation is proved recalling that, for ℓ odd, the function $c \mapsto E_{n\ell}(c)$ is odd as well; this implies that the general term of the sum (B.25) changes its sign under a transformation $h \mapsto -h$.
Finally, let us prove (B.28) for any even ℓ. In this case we have an even polynomial

$$E_{n\ell}(c) = \sum_{j=0,2,\dots,\ell+2} E_{n\ell j} c^j \ , \tag{B.29}$$

so (B.25) implies

$$\mathcal{Q}_{n\ell}(u) = 2 \sum_{j=0,2,\dots,\ell+2} E_{n\ell j} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(\widehat{h}\bullet u)^j}{|h|^{2n-\ell}} \ ; \tag{B.30}$$

in particular, for the $j = 2$ term above we have

$$\sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(\widehat{h}\bullet u)^2}{|h|^{2n-\ell}} = \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(h\bullet u)^2}{|h|^{2n-\ell+2}} = \frac{1}{d} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{1}{|h|^{2n-\ell}} \ , \tag{B.31}$$

where the last passage follows from the identity (A.7) (with k replaced by u and $\varphi(|h|) = 1/|h|^{2n-\ell+2}$). Eqs. (B.30) (B.31) imply

$$\mathcal{Q}_{n\ell}(u) = 2 \sum_{j=0,4,6,\dots,\ell+2} E_{n\ell j} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{(\widehat{h}\bullet u)^j}{|h|^{2n-\ell}} + \frac{2E_{n\ell 2}}{d} \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{1}{|h|^{2n-\ell}} \ . \tag{B.32}$$

On the other hand, the Definition A.9 of $\hat{E}_{n\ell}$ prescribes

$$\hat{E}_{n\ell}(c) = \sum_{j=0,4,6,\dots,\ell+2} E_{n\ell j} c^j + \frac{E_{n\ell 2}}{d} \ ; \tag{B.33}$$

comparing this with (B.32), we conclude

$$\mathcal{Q}_{n\ell}(u) = 2 \sum_{h \in \mathbf{Z}^d_0, |h|<\rho} \frac{\hat{E}_{n\ell}(\widehat{h}\bullet u)}{|h|^{2n-\ell}} \quad \text{as in (B.10)} \ ,$$

and (B.28) is proved.
Step 7. Let $t \in \{2, 4, 6, ...\}$. For $\ell \in \{2, 4, ..., t-2\}$, the $\mathcal{Q}_{n\ell}$ are polynomial functions on $\mathbf{S}^{d-1}$; considering their minima $q_{n\ell}$ and maxima $Q_{n\ell}$, one infers from (B.7) the inequalities (B.12)

$$Z_n + \sum_{\ell=2,4,\dots,t-2} \frac{q_{n\ell}}{|k|^\ell} + \frac{v_{nt}}{|k|^t} \leqslant \mathcal{K}_n(k) \leqslant Z_n + \sum_{\ell=2,4,\dots,t-2} \frac{Q_{n\ell}}{|k|^\ell} + \frac{V_{nt}}{|k|^t} \quad \textit{for } |k| \geqslant 2\rho \ ,$$

which imply the relation (B.13) $\mathcal{K}_n(k) \to Z_n$ for $k \to \infty$. The polynomial nature of each function $\mathcal{Q}_{n\ell}$ follows from its definition (B.10) in terms of the polynomial $\hat{E}_{n\ell}$. The inequalities (B.12) for $\mathcal{K}_n(k)$ are obvious; the statement (B.13) follows noting that, in Eq. (B.12), both the lower and the upper bound for $\mathcal{K}_n(k)$ tend to Z_n for $k \to \infty$.

Step 8. Proof of the inequalities (B.14)

$$\sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_n(k) \leqslant \sup_{k \in \mathbf{Z}_0^d} \mathscr{K}_n(k) \leqslant \Big(\sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_n(k) \Big) + \delta\mathcal{K}_n < +\infty \ .$$

The first two of the above inequalities are an obvious consequence of the relations (B.2) $\mathcal{K}_n(k) < \mathscr{K}_n(k) \leqslant \mathcal{K}_n(k) + \delta\mathcal{K}_n$; the third inequality holds if we show that

$$\sup_{k \in \mathbf{Z}_0^d} \mathcal{K}_n(k) < +\infty \ , \tag{B.34}$$

and this follows from the existence of a finite $k \to \infty$ limit for $\mathcal{K}_n(k)$ (see Step 7). $\square$

C Appendix. The upper bounds K_n^+, for $d = 3$ and $n = 2, 3, 4, 5, 10$

Eq. (3.20) defines K_n^+ in terms of $\sup_{k \in \mathbf{Z}_0^3} \mathscr{K}_n(k)$, or of any upper approximant for this sup. In all the cases analyzed hereafter, we produce both an upper and a lower approximant; the lower one is given only to indicate the uncertainty in our evaluation of $\sup \mathscr{K}_n$.

Some details on the evaluation of $\mathscr{K}_2$ and of its sup. Among the examples presented here, the case of $\mathscr{K}_2$ is the one requiring more expensive computations. To evaluate $\mathscr{K}_2$, we apply Proposition B.1 with a fairly large cutoff

$$\rho = 20 \ ; \tag{C.1}$$

thus, we must often sum over the set $\{h \in \mathbf{Z}_0^3 \mid |h| < 20\}$. Eq. (B.5) gives

$$\delta\mathcal{K}_2 = 5.6856... \ , \tag{C.2}$$

and it remains to evaluate the function $\mathcal{K}_2$, using directly the definition (B.4) or the bounds in Proposition B.1.

To evaluate $\mathcal{K}_2(k)$, we start from the $k's$ in $\mathbf{Z}_0^3$ with $|k| < 2\rho = 40$. We use directly the definition (B.4) for all such k's ([2]); in this way, we obtain

$$\max_{k \in \mathbf{Z}_0^3, |k| < 40} \mathcal{K}_2(k) = \mathcal{K}_2(9, 9, 9) = 22.022... \tag{C.3}$$

[2] In fact, due to the symmetry properties (B.6), computation of $\mathcal{K}_2(k)$ can be limited to points k such that $k_1 \geqslant k_2 \geqslant k_3 \geqslant 0$.

(another result is that $\mathcal{K}_2(k)$ has a small oscillation for $|k|$ between 10 and 40, since $\mathcal{K}_2(k) \geqslant 21.563$ for $k \in \mathbf{Z}^3$, $10 < |k| < 40$).

Let us pass to the case $|k| \geqslant 40$. Here, our main tool is the upper bound (B.12) with $t = 6$; after some computation, this yields the result ([3])

$$\mathcal{K}_2(k) \leqslant 21.205 + \frac{598.28}{|k|^2} + \frac{1.1507 \times 10^5}{|k|^4} + \frac{1.1795 \times 10^9}{|k|^6} \leqslant 21.912\,. \tag{C.4}$$

(For completeness, we also mention that the $t = 6$ lower bound in (B.12) and Eq. (B.13) imply $\inf_{k \in \mathbf{Z}^d_0, |k| \geqslant 40} \mathcal{K}_2(k) = \lim_{k \to \infty} \mathcal{K}_2(k) = 21.204...$ ([4]); by comparison with (C.4), we see that $\mathcal{K}_2(k)$ is almost constant for $|k| \geqslant 40$.)

The results (C.3) (C.4) yield

$$\sup_{k \in \mathbf{Z}^3_0} \mathcal{K}_2(k) = \mathcal{K}_2(9, 9, 9) = 22.022... \,. \tag{C.5}$$

[3] Let us give some supplementary information on the computations yielding (C.4). The $t = 6$ upper bound in (B.12) reads

$$\mathcal{K}_2(k) \leqslant Z_2 + \frac{Q_{22}}{|k|^2} + \frac{Q_{24}}{|k|^4} + \frac{V_{26}}{|k|^6} \ ,$$

and we must determine the constants Z_2, etc., appearing therein. Z_2 and V_{26} are computed directly from the definitions (B.8) (B.9) (the second one requiring previous knowledge of $M_{26} = 73.835...$, see Eq. (A.19)); in this way one gets $Z_2 = 21.204...$ and $V_{26} = 1.1794... \times 10^9$. Q_{22} is the maximum of the polynomial function $\mathcal{Q}_{22} : \mathbf{S}^2 \to \mathbf{R}$, and Eq. (B.10) gives for this function the explicit expression

$$\mathcal{Q}_{22}(u) = 2904.7... - 4569.7...\,(u_1^2 u_2^2 + u_1^2 u_3^2 + u_2^2 u_3^2) - 2349.8...\,(u_1^4 + u_2^4 + u_3^4) \ ,$$

for all $u \in \mathbf{S}^2$; one finds $Q_{22} = \mathcal{Q}_{22}(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = 598.27...\,$.

Q_{24} is the maximum of the polynomial function $\mathcal{Q}_{24} : \mathbf{S}^2 \to \mathbf{R}$; after computing this function via Eq. (B.10), one gets $Q_{24} = 1.1506... \times 10^5$.

After rounding up from above all the numerical outputs, the computation we have just outlined gives the first inequality (C.4) $\mathcal{K}_2(k) \leqslant 21.205 + 598.28|k|^{-2} +$ etc., holding for $|k| \geqslant 40$; on the other hand, $21.205 + 598.28|k|^{-2} +$ etc. $\leqslant 21.912$ for all such k's, which explains the second inequality (C.4)

[4] First of all, Eq. (B.13) gives

$$\lim_{k \to \infty} \mathcal{K}_2(k) = Z_2$$

where Z_2 is defined by (B.8); from the previous footnote, we know that $Z_2 = 21.204....$ Now, let us pass to the lower bound (B.12), with $t = 6$; after computing all the necessary constants and rounding up the results from below, we obtain

$$\mathcal{K}_2(k) \geqslant Z_2 + \frac{554.98}{|k|^2} + \frac{1.1413 \times 10^5}{|k|^4} - \frac{3.6293 \times 10^8}{|k|^6} \qquad \text{for } k \in \mathbf{Z}^3_0,\ |k| \geqslant 40.$$

On the other hand, one has $554.98\,|k|^{-2} + 1.1413 \times 10^5\,|k|^{-4}... \geqslant 0$ for $|k| \geqslant 40$; so, $\inf_{k \in \mathbf{Z}^3_0, |k| \geqslant 40} \mathcal{K}_2(k) \geqslant Z_2$. It is obvious that $\inf_{k \in \mathbf{Z}^3_0, |k| \geqslant 40} \mathcal{K}_2(k) \leqslant \lim_{k \in \mathbf{Z}^3_0, k \to \infty} \mathcal{K}_2(k)$; the latter equals Z_2, thus $\inf_{|k| \geqslant 40} \mathcal{K}_2(k) = \lim_{k \to \infty} \mathcal{K}_2(k) = Z_2$.

We now pass to the function $\mathcal{K}_2$; according to (B.14) we have $\sup_{k\in\mathbf{Z}_0^3}\mathcal{K}_2(k) \leqslant \sup_{k\in\mathbf{Z}_0^3}\mathcal{K}_2(k) \leqslant \Big(\sup_{k\in\mathbf{Z}_0^3}\mathcal{K}_2(k)\Big) + \delta\mathcal{K}_2$, and the results (C.2) (C.5) give

$$22.022 < \sup_{k\in\mathbf{Z}_0^3}\mathcal{K}_2(k) < 27.709 \ . \tag{C.6}$$

(The uncertainty on this sup is fairly large, due to the value of $\delta\mathcal{K}_2$ in (C.2); the error $\delta\mathcal{K}_2$ could be significantly reduced choosing a cutoff $\rho \gg 20$, but the related computations would be much more expensive.)

The upper bound $\boldsymbol{K_2^+}$. According to the definition (3.20), we have

$$K_2^+ = \frac{1}{(2\pi)^{3/2}}\sqrt{\sup_{k\in\mathbf{Z}_0^3}\mathcal{K}_2(k)} \quad \text{(or any upper approximant for this)} \ . \tag{C.7}$$

Due to (C.6), we can take $K_2^+ = (2\pi)^{-3/2}\sqrt{27.709}$; rounding up to three digits we can write

$$K_2^+ = 0.335\,, \tag{C.8}$$

as reported in (3.23).

Preparing the examples with $\boldsymbol{n = 3, 4, 5, 10}$. To evaluate $\mathcal{K}_n$ for the cited values of n, we apply Proposition B.1 with a cutoff

$$\rho = 10 \ ; \tag{C.9}$$

thus, all sums over h in Proposition B.1 are over the set $\{h \in \mathbf{Z}_0^3 \mid |h| < 10\}$.

Some details on the evaluation of $\boldsymbol{\mathcal{K}_3}$ and of its sup. Eq. (B.5) gives

$$\delta\mathcal{K}_3 = 0.45295... \ , \tag{C.10}$$

and it remains to evaluate the function $\mathcal{K}_3$. Direct computation of this function from the definition (B.4), for all k's of norm < 20, gives

$$\max_{k\in\mathbf{Z}_0^3, |k|<20}\mathcal{K}_3(k) = \mathcal{K}_3(2,1,1) = 25.301... \ . \tag{C.11}$$

On the other hand the upper bound in Eq. (B.12), with $t = 6$, gives

$$\mathcal{K}_3(k) \leqslant 11.197 + \frac{117.33}{|k|^2} + \frac{1581.5}{|k|^4} + \frac{3.3994\times 10^6}{|k|^6} \leqslant 11.554$$
$$\text{for } k \in \mathbf{Z}_0^3,\ |k| \geqslant 20 \ . \tag{C.12}$$

(For completeness, we mention that the $t = 6$ lower bound in (B.12) and Eq. (B.13) give $\inf_{k\in\mathbf{Z}_0^3,|k|\geqslant 20}\mathcal{K}_3(k) = \lim_{k\to\infty}\mathcal{K}_3(k) = 11.196...$. Comparing with (C.12) we conclude that $\mathcal{K}_3(k)$ is almost constant for $|k| \geqslant 20$.)

Eqs. (C.11) (C.12) yield

$$\sup_{k \in \mathbf{Z}_0^3} \mathcal{K}_3(k) = \mathcal{K}_3(2,1,1) = 25.301... \ . \tag{C.13}$$

We now pass to the function $\mathcal{K}_3$; according to (B.14) we have $\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_3(k) \leqslant \sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_3(k) \leqslant \Big(\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_3(k)\Big) + \delta\mathcal{K}_3$, and the results (C.10) (C.13) give

$$25.301 < \sup_{k \in \mathbf{Z}_0^3} \mathcal{K}_2(k) < 25.755 \ . \tag{C.14}$$

The upper bound K_3^+. According to the definition (3.20), we have

$$K_3^+ = \frac{1}{(2\pi)^{3/2}} \sqrt{\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_3(k)} \quad \text{(or any upper approximant for this)} \ . \tag{C.15}$$

Due to (C.14), we can take $K_3^+ = (2\pi)^{-3/2}\sqrt{25.755}$; rounding up to three digits we can write

$$K_3^+ = 0.323 \, , \tag{C.16}$$

as reported in (3.23).

Some details on the evaluation of $\mathcal{K}_4$ and of its sup. Eq. (B.5) gives

$$\delta\mathcal{K}_4 = 0.021561... \ , \tag{C.17}$$

and it remains to evaluate the function $\mathcal{K}_4$. Direct computation of this function from the definition (B.4), for all k's of norm < 20, gives

$$\max_{k\in\mathbf{Z}_0^3, |k|<20} \mathcal{K}_4(k) = \mathcal{K}_4(2,1,0) = 48.038... \ . \tag{C.18}$$

On the other hand the upper bound in Eq. (B.12), with $t = 6$, gives

$$\mathcal{K}_4(k) \leqslant 9.2611 + \frac{137.37}{|k|^2} + \frac{629.55}{|k|^4} + \frac{4.2612 \times 10^5}{|k|^6} \leqslant 9.6152 \tag{C.19}$$

$$\text{for } k \in \mathbf{Z}_0^3, \ k \geqslant 20 \ .$$

(For completeness we mention that the $t = 6$ lower bound in (B.12) implies $\mathcal{K}_4(k) \geqslant 9.2380$ for $k \in \mathbf{Z}_0^3$, $|k| \geqslant 20$ ([5]), while Eq. (B.13) gives $\lim_{k\to\infty} \mathcal{K}_4(k) = 9.2610\ldots$.)

[5]More precisely: the $t = 6$ lower bound in (B.12) gives $\mathcal{K}_4(k) \geqslant 9.2610 - 10.098\,|k|^{-2} + 446.33\,|k|^{-4} - 3.1595 \times 10^4\,|k|^{-6}$ for $k \in \mathbf{Z}_0^3$, $|k| \geqslant 20$; for k in the same range, one has $9.2610 - 10.098\,|k|^{-2} + ... \geqslant 9.2380$.

Eqs. (C.18) (C.19) yield

$$\sup_{k \in \mathbf{Z}^3_0} \mathcal{K}_4(k) = \mathcal{K}_4(2,1,0) = 48.038.. \; . \tag{C.20}$$

We now pass to the function $\mathscr{K}_4$; according to (B.14) we have $\sup_{k \in \mathbf{Z}^3_0} \mathcal{K}_4(k) \leqslant \sup_{k \in \mathbf{Z}^3_0} \mathscr{K}_4(k) \leqslant \Big(\sup_{k \in \mathbf{Z}^3_0} \mathcal{K}_4(k) \Big) + \delta\mathcal{K}_4$, and the results (C.17) (C.20) give

$$48.038 < \sup_{k \in \mathbf{Z}^3_0} \mathscr{K}_4(k) < 48.061 \; . \tag{C.21}$$

The upper bound $\boldsymbol{K_4^+}$. According to the definition (3.20), we have

$$K_4^+ = \frac{1}{(2\pi)^{3/2}} \sqrt{\sup_{k \in \mathbf{Z}^3_0} \mathscr{K}_4(k)} \quad \text{(or any upper approximant for this)} \; . \tag{C.22}$$

Due to (C.21), we can take $K_4^+ = (2\pi)^{-3/2}\sqrt{48.061}$; rounding up to three digits we can write

$$K_4^+ = 0.441 \, , \tag{C.23}$$

as reported in (3.23).

Some details on the evaluation of $\boldsymbol{\mathscr{K}_5}$ and of its sup. Eq. (B.5) gives

$$\delta\mathcal{K}_5 = 0.0012414... \; , \tag{C.24}$$

and it remains to evaluate the function $\mathcal{K}_5$. Direct computation of this function from the definition (B.4), for all k's of norm < 20, gives

$$\max_{k \in \mathbf{Z}^3_0, |k| < 30} \mathcal{K}_5(k) = \mathcal{K}_5(1,1,0) = 64.455... \; . \tag{C.25}$$

On the other hand the upper bound in Eq. (B.12), with $t = 6$, gives

$$\mathcal{K}_5(k) \leqslant 8.5682 + \frac{186.23.}{|k|^2} + \frac{919.89}{|k|^4} + \frac{2.2152 \times 10^5}{|k|^6} \leqslant 9.0430$$
$$\text{for } k \in \mathbf{Z}^3_0, \; k \geqslant 20 \; . \tag{C.26}$$

(For completeness we mention that the $t = 6$ lower bound in (B.12) implies $\mathcal{K}_5(k) \geqslant 8.4974$ for $k \in \mathbf{Z}^3_0$, $|k| \geqslant 20$, while Eq. (B.13) gives $\lim_{k \to \infty} \mathcal{K}_5(k) = 8.5681...$.)
Eqs. (C.25) (C.26) yield

$$\sup_{k \in \mathbf{Z}^3_0} \mathcal{K}_5(k) = \mathcal{K}_5(1,1,0) = 64.455... \; . \tag{C.27}$$

We now pass to the function $\mathcal{K}_5$; according to (B.14) we have $\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_5(k) \leqslant \sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_5(k) \leqslant \Big(\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_5(k)\Big) + \delta\mathcal{K}_5$, and the results (C.24) (C.27) give

$$64.455 < \sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_5(k) < 64.458\ . \tag{C.28}$$

The upper bound K_5^+. According to the definition (3.20), we have

$$K_5^+ = \frac{1}{(2\pi)^{3/2}}\sqrt{\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_5(k)} \quad \text{(or any upper approximant for this)}\ . \tag{C.29}$$

Due to (C.28), we can take $K_5^+ = (2\pi)^{-3/2}\sqrt{64.458}$; rounding up to three digits we can write

$$K_5^+ = 0.510\,, \tag{C.30}$$

as reported in (3.23).

Some details on the evaluation of $\mathcal{K}_{10}$ and of its sup. Eq. (B.5) gives

$$\delta\mathcal{K}_{10} = 2.1401... \times 10^{-9}\ , \tag{C.31}$$

and it remains to evaluate the function $\mathcal{K}_{10}$. Direct computation of this function from the definition (B.4), for all k's of norm < 20, gives

$$\max_{k\in\mathbf{Z}_0^3, |k|<20} \mathcal{K}_{10}(k) = \mathcal{K}_{10}(1,1,0) = 2048.0...\,. \tag{C.32}$$

On the other hand the upper bound in Eq. (B.12), with $t = 6$, gives

$$\mathcal{K}_{10}(k) \leqslant 8.0159 + \frac{617.05}{|k|^2} + \frac{9693.2}{|k|^4} + \frac{5.6557\times 10^7}{|k|^6} \leqslant 10.503 \tag{C.33}$$

$$\text{for } k \in \mathbf{Z}_0^3,\ k \geqslant 20\ .$$

(For completeness we mention that the $t = 6$ lower bound in (B.12) implies $\mathcal{K}_{10}(k) \geqslant 7.8034$ for $k \in \mathbf{Z}_0^3$, $|k| \geqslant 20$, while (B.13) gives $\lim_{k\to\infty} \mathcal{K}_{10}(k) = 8.0158...\,$.)
Eqs. (C.32) (C.33) yield

$$\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_{10}(k) = \mathcal{K}_{10}(1,1,0) = 2048.0...\ . \tag{C.34}$$

We now pass to the function $\mathcal{K}_{10}$; according to (B.14) we have $\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_{10}(k) \leqslant \sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_{10}(k) \leqslant \Big(\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_{10}(k)\Big) + \delta\mathcal{K}_{10}$, and the results (C.31) (C.34) give ([6])

$$\sup_{k\in\mathbf{Z}_0^3} \mathcal{K}_{10}(k) = 2048.0...\ . \tag{C.35}$$

[6] In the MATHEMATICA output for $\mathcal{K}_{10}(1,1,0)$, 2048.0 is followed by a digit different from 9; so, the digits 2048.0 do not change when $\delta\mathcal{K}_{10}$ is added to this output.

The upper bound K^+_{10}. According to the definition (3.20), we have

$$K^+_{10} = \frac{1}{(2\pi)^{3/2}} \sqrt{\sup_{k \in \mathbf{Z}^3_0} \mathcal{K}_{10}(k)} \quad \text{(or any upper approximant for this)} . \tag{C.36}$$

Using (C.35), and rounding up to three digits the final result, we can write

$$K^+_{10} = 2.88 \,, \tag{C.37}$$

as reported in (3.23).

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